ECE 222b
Applied Electromagnetics
Notes Set 1

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Maxwell's Equations (1)

- Ampere (1826) $\oint_C \vec{B} \cdot d\vec{l} = \mu_0 \iint_S \vec{J} \cdot d\vec{s}$
- Faraday (1831) $\oint_c \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint_S \vec{B} \cdot d\vec{s}$
- Maxwell (1860) $\oint_C \vec{B} \cdot d\vec{l} = \mu_0 \varepsilon_0 \frac{d}{dt} \iint_S \vec{E} \cdot d\vec{s} + \mu_0 \iint_S \vec{J} \cdot d\vec{s}$
- Gauss (1832) $\oiint_S \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon_0}$
- Lorentz (1900) $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

Maxwell's Equations (2)

Fundamental postulates **(in terms of total currents and charges):**

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint_S \vec{B} \cdot d\vec{s}$$ **Faraday's law**

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 \varepsilon_0 \frac{d}{dt} \iint_S \vec{E} \cdot d\vec{s} + \mu_0 I$$ **Maxwell-Ampere's law**

$$\oiint_S \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon_0}$$ **Gauss's law**

$$\oiint_S \vec{B} \cdot d\vec{s} = 0$$ **Gauss's law-magnetic**

Maxwell's equations in integral form

Basic quantities:

$\vec{E}$: V/m B: webers/m^2

I: A Q: coulumbs

Maxwell's Equations (3)

For continuous field **E**,

$$\because \oint_C \vec{E} \cdot d\vec{l} = \iint_S \nabla \times \vec{E} \cdot d\vec{s}$$ Stokes's theorem

$$\therefore \iint_S \nabla \times \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \iint_S \vec{B} \cdot d\vec{s} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$ Faraday's law

Similarly,

$$\boxed{\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}}$$ Maxwell-Ampere's law

Maxwell’s Equations (4)

For continuous field **E**,

$$\because \oiint_S \vec{E} \cdot d\vec{s} = \iiint_V \nabla \cdot \vec{E}\, dv$$

Gauss’s theorem

$$\therefore \iiint_V \nabla \cdot \vec{E}\, dv = \frac{1}{\varepsilon_0} \iiint_V \rho\, dv$$

$$\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}}$$

Gauss’s law

Similarly,

$$\boxed{\nabla \cdot \vec{B} = 0}$$

Gauss’s law - magnetic

Maxwell's Equations (5)

Maxwell's equations in differential form:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ **Faraday's law**

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$$ **Maxwell-Ampere's law**

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$ **Gauss's law**

$$\nabla \cdot \vec{B} = 0$$ **Gauss's law-magnetic**

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*******Valid for total currents and charges*******
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Maxwell's Equations (6)

Maxwell-Ampere's law:

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$$

$$\nabla \cdot \left(\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J} \right) \quad \Rightarrow \quad 0 = \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J}$$

$$\boxed{\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}}$$

$$\iiint_V \nabla \cdot \vec{J}\, dv = -\iiint_V \frac{\partial \rho}{\partial t} dv$$

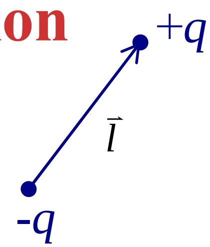
Continuity equation

$$\boxed{\oiint_S \vec{J} \cdot d\vec{s} = -\frac{dQ}{dt}}$$

Constitutive Relations (1)

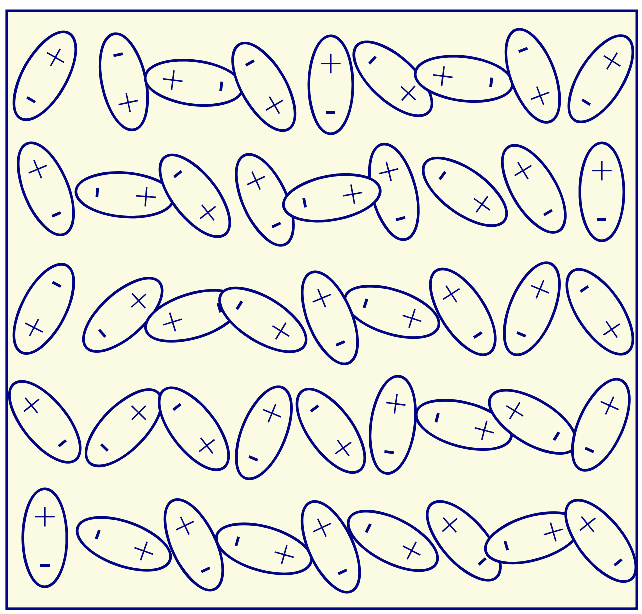
1. Electric polarization

Electric dipole:

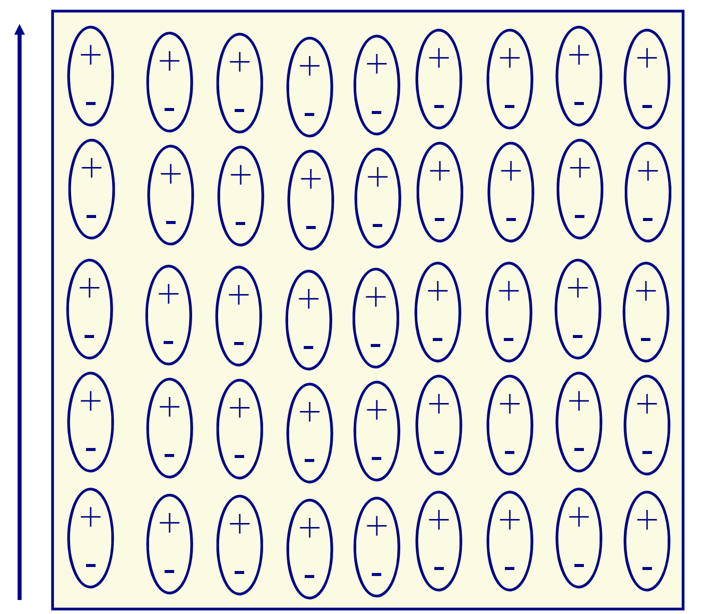


$\vec{p} = q\vec{l}$ ⇐ Dipole moment

No applied field



With an applied field **E**



Constitutive Relations (2)

Net polarization: $$\vec{P} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n_p} \vec{p}_i$$

Electric polarization vector

Bound charge density: $$\rho_b = -\nabla \cdot \vec{P}$$

Gauss' s law: $$\nabla \cdot \vec{E} = \frac{\rho_{\text{total}}}{\varepsilon_0} = \frac{\rho_{\text{free}} + \rho_b}{\varepsilon_0} = \frac{\rho_{\text{free}} - \nabla \cdot \vec{P}}{\varepsilon_0}$$

$$\nabla \cdot (\varepsilon_0 \vec{E} + \vec{P}) = \rho_{\text{free}} \quad \Longrightarrow \quad \nabla \cdot \vec{D} = \rho_{\text{free}}$$

Define electric flux density: $$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

Constitutive Relations (3)

In simple matter:

$$\vec{P} \sim \vec{E} \quad \text{or} \quad \vec{P} = \varepsilon_0 \chi_e \vec{E}$$

Electric susceptibility (χ_e)

$$\boxed{\vec{D} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E} = \varepsilon \vec{E}}$$

$$\varepsilon = \varepsilon_0 + \varepsilon_0 \chi_e$$ ⇐ Permittivity

Relative permittivity: $\varepsilon_r = \dfrac{\varepsilon}{\varepsilon_0} = 1 + \chi_e$

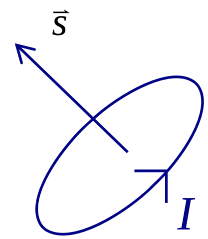
Polarization current:

$$\nabla \cdot \vec{J}_p = -\frac{\partial \rho_b}{\partial t} \quad \Longrightarrow \quad \vec{J}_p = \frac{\partial \vec{P}}{\partial t}$$

Constitutive Relations (4)

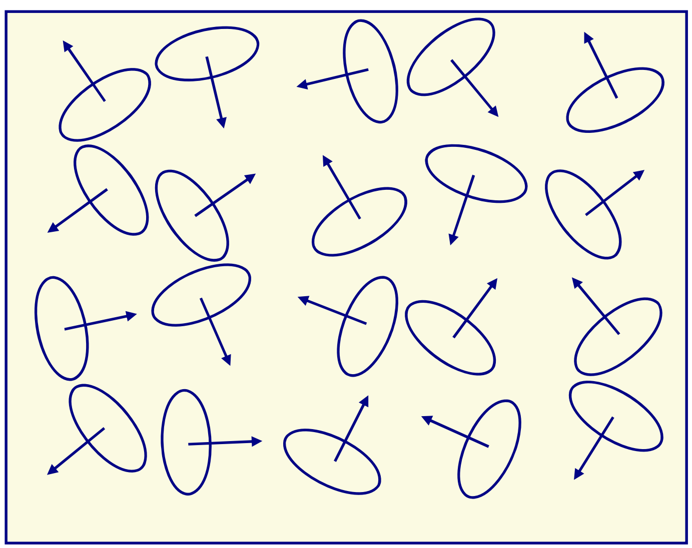
2. Magnetization

Magnetic dipole:

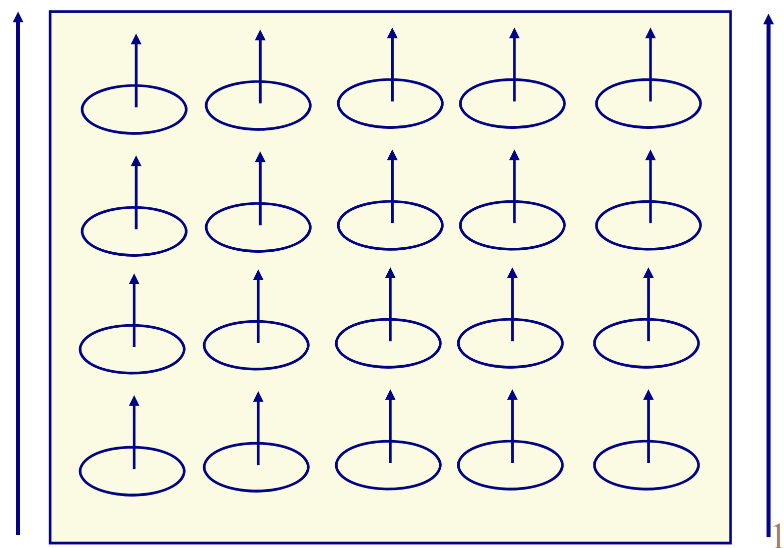


$\vec{m} = I\vec{s}$ ⇐ Dipole moment

No applied field



With an applied field **B**



Constitutive Relations (5)

Applied field **B** creates net magnetic dipole moment:

$$\vec{\mathcal{M}} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n_m} \vec{m}_i$$

⟵ magnetization vector

Magnetization current: $\vec{J}_m = \nabla \times \vec{\mathcal{M}}$

Total current: $\vec{J}_{\text{total}} = \vec{J}_p + \vec{J}_m + \vec{J}_{\text{free}} = \dfrac{\partial \vec{P}}{\partial t} + \nabla \times \vec{\mathcal{M}} + \vec{J}_{\text{free}}$

Maxwell-Ampere's law:

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}_{\text{total}} \quad \Rightarrow \quad \nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{\mathcal{M}} \right) = \frac{\partial \vec{D}}{\partial t} + \vec{J}_{\text{free}}$$

Constitutive Relations (6)

Define magnetic field intensity:

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{\mathcal{M}} \qquad \vec{B} = \mu_0 \vec{H} + \mu_0 \vec{\mathcal{M}} \quad \Longrightarrow \quad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}_{\text{free}}$$

In simple matter:

$$\vec{\mathcal{M}} \sim \vec{H} \quad \text{or} \quad \vec{\mathcal{M}} = \chi_m \vec{H}$$

↖ Magnetic susceptibility

$$\boxed{\vec{B} = \mu_0 \vec{H} + \mu_0 \chi_m \vec{H} = \mu \vec{H}}$$

$$\mu = \mu_0 + \mu_0 \chi_m \quad \Longleftarrow \quad \text{Permeability}$$

Relative permeability: $\mu_r = \dfrac{\mu}{\mu_0} = 1 + \chi_m$

Constitutive Relations (7)

3. Conduction ⇨ Electric conduction current

$$\vec{J}_c = \sigma \vec{E}$$

σ ⟵ Conductivity (s/m)

4. Classification of material

a. Based on space variable

Homogeneous vs Inhomogeneous

b. Based on time variable

Stationary vs Nonstationary

c. Based on the value of conductivity

Perfect dielectric vs Perfect conductor

Constitutive Relations (8)

4. Classification of material (continued)

d. Based on the direction of **D** and **B**

If $\vec{D} \parallel \vec{E}, \vec{B} \parallel \vec{H} \quad \Rightarrow$ Isotropic

$$\vec{D} = \varepsilon \vec{E}: \quad D_x = \varepsilon E_x, D_y = \varepsilon E_y, D_z = \varepsilon E_z$$

If $\vec{D} \not\parallel \vec{E}, \vec{B} \not\parallel \vec{H} \quad \Rightarrow$ Anisotropic

$$D_x = \varepsilon_{xx} E_x + \varepsilon_{xy} E_y + \varepsilon_{xz} E_z$$
$$D_y = \varepsilon_{yx} E_x + \varepsilon_{yy} E_y + \varepsilon_{yz} E_z$$
$$D_z = \varepsilon_{zx} E_x + \varepsilon_{zy} E_y + \varepsilon_{zz} E_z$$

$$\Rightarrow \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

More compact: $\vec{D} = \overline{\overline{\varepsilon}} \cdot \vec{E} \qquad \vec{B} = \overline{\overline{\mu}} \cdot \vec{H}$

tensor

Constitutive Relations (9)

For crystals:

$$\overline{\overline{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix}$$

If $\varepsilon_{xx} \neq \varepsilon_{yy} \neq \varepsilon_{zz}$ ⟹ Biaxial

If $\varepsilon_{xx} = \varepsilon_{yy} \neq \varepsilon_{zz}$ ⟹ Uniaxial

Isotropic medium: ε, μ are scalars

Anisotropic medium: $\overline{\overline{\varepsilon}}, \overline{\overline{\mu}}$ are (non - identity) tensors

Bianisotrpic medium:

$$\vec{D} = \overline{\overline{\varepsilon}} \cdot \vec{E} + \overline{\overline{\xi}} \cdot \vec{H}$$

$$\vec{B} = \overline{\overline{\mu}} \cdot \vec{H} + \overline{\overline{\zeta}} \cdot \vec{E}$$

Constitutive Relations (10)

4. Classification of material (continued)

e. Based on the form of ε, μ, σ

Linear vs Nonlinear

↑

$$\varepsilon = \varepsilon(\vec{E}, \vec{H}), \quad \mu = \mu(\vec{E}, \vec{H}), \quad \sigma = \sigma(\vec{E}, \vec{H})$$

f. Based on the relation of ε and μ with frequency

Dispersive vs. Nondispersive

↑

$$\varepsilon = \varepsilon(f), \quad \mu = \mu(f)$$

Maxwell's Equations (7)

Fundamental postulates **(in terms of free currents and charges):**

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\iint_S \vec{B} \cdot d\vec{s} - K$$ **Faraday's law**

$$\oint_C \vec{H} \cdot d\vec{l} = \frac{d}{dt}\iint_S \vec{D} \cdot d\vec{s} + I$$ **Maxwell-Ampere's law**

$$\oint\!\!\oint_S \vec{D} \cdot d\vec{s} = Q$$ **Gauss's law**

$$\oint\!\!\oint_S \vec{B} \cdot d\vec{s} = Q_m$$ **Gauss's law-magnetic**

Maxwell's equations in integral form

Basic quantities:

$\vec{E}$: V/m $\vec{H}$: A/m

$\vec{D}$: C/m^2 $\vec{B}$: webers/m^2

K : V I : A

Q : coulumbs Q_m : webers

Maxwell's Equations (8)

Maxwell's equations in differential form:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \vec{M}$$ **Faraday's law**

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$ **Maxwell-Ampere's law**

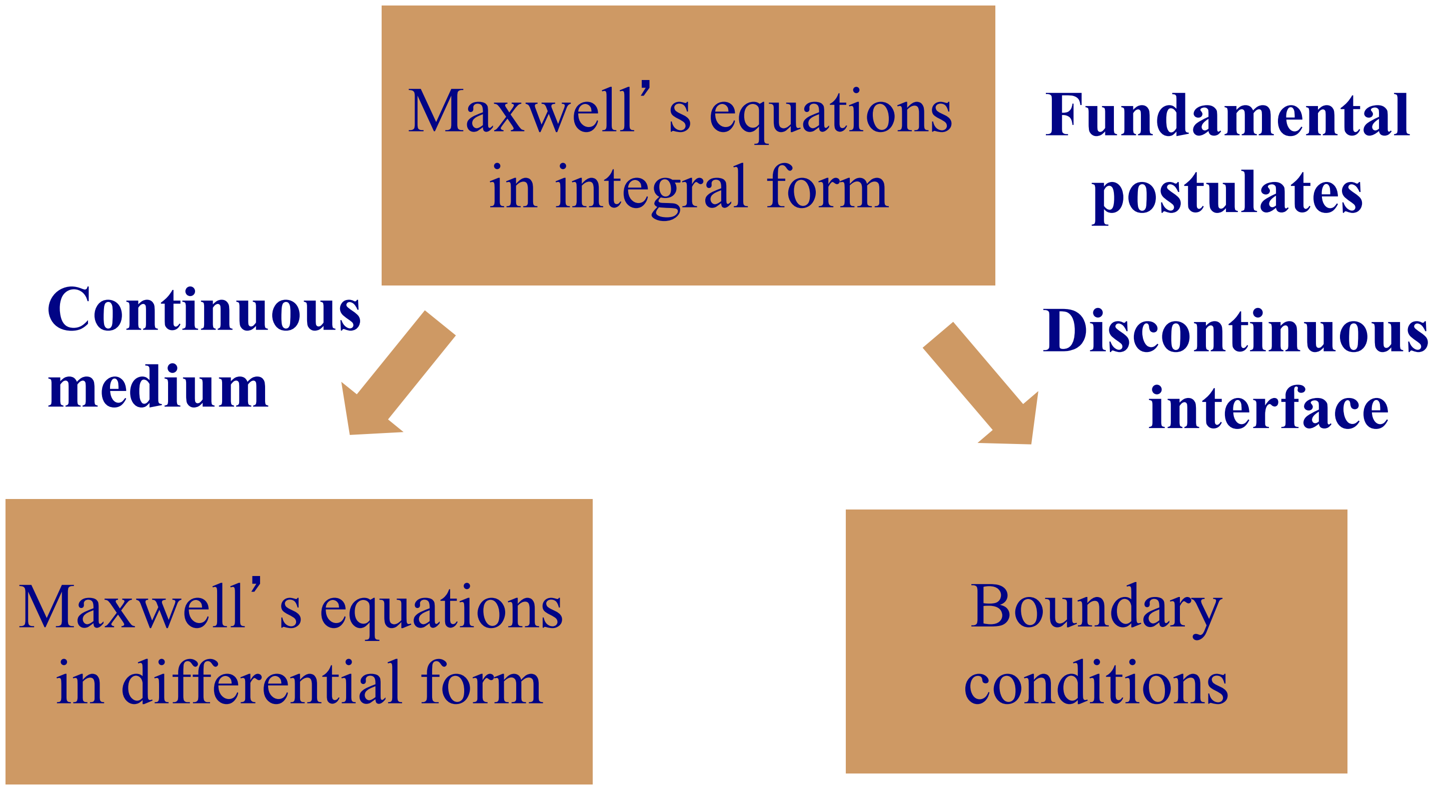
$$\nabla \cdot \vec{D} = \rho$$ **Gauss's law**

$$\nabla \cdot \vec{B} = \rho_m$$ **Gauss's law-magnetic**

Constitutive relations:

$$\vec{D} = \varepsilon \vec{E} \qquad \vec{B} = \mu \vec{H} \qquad \vec{J}_c = \sigma \vec{E}$$

Boundary Conditions (0)



Boundary Conditions (5)

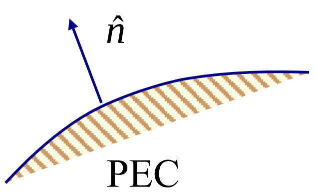
Boundary conditions:

$$\hat{n}\times(\vec{E}_2-\vec{E}_1)=0 \qquad (\text{or} \quad -\vec{M}_s)$$
$$\hat{n}\times(\vec{H}_2-\vec{H}_1)=\vec{J}_s$$
$$\hat{n}\cdot(\vec{D}_2-\vec{D}_1)=\rho_s$$
$$\hat{n}\cdot(\vec{B}_2-\vec{B}_1)=0 \qquad (\text{or} \quad \rho_{ms})$$

If medium 1 is a perfect conductor, then

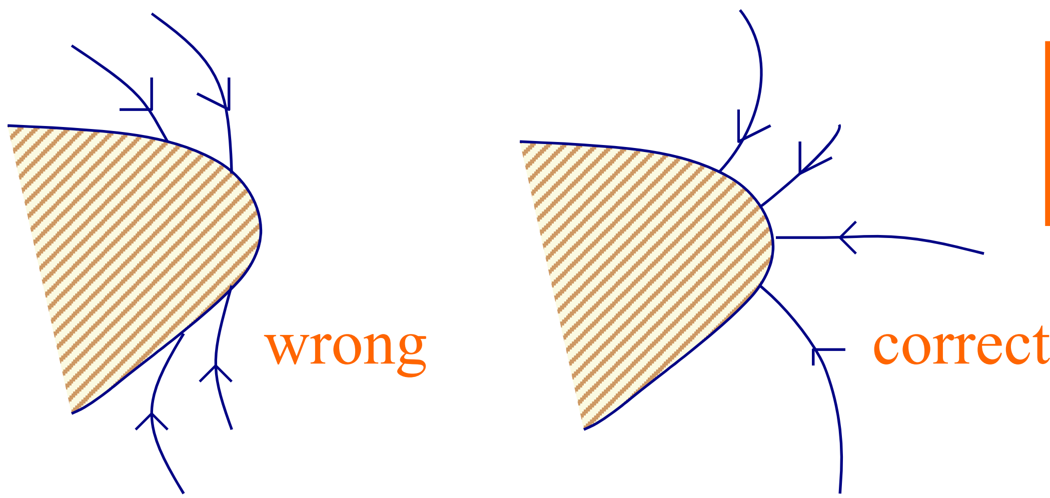
$$\vec{E}_1=\vec{H}_1=\vec{D}_1=\vec{B}_1=0$$

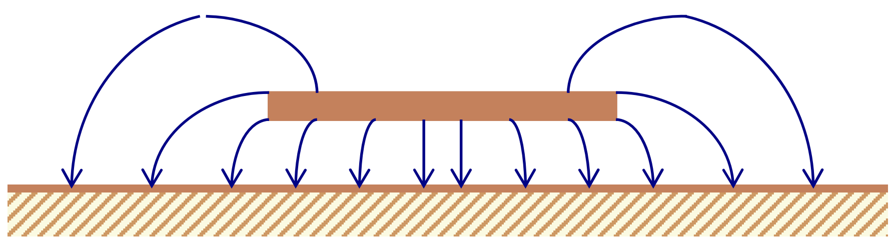
$$\hat{n}\times\vec{E}=0 \qquad \hat{n}\cdot\vec{D}=\rho_s$$
$$\hat{n}\times\vec{H}=\vec{J}_s \qquad \hat{n}\cdot\vec{B}=0$$



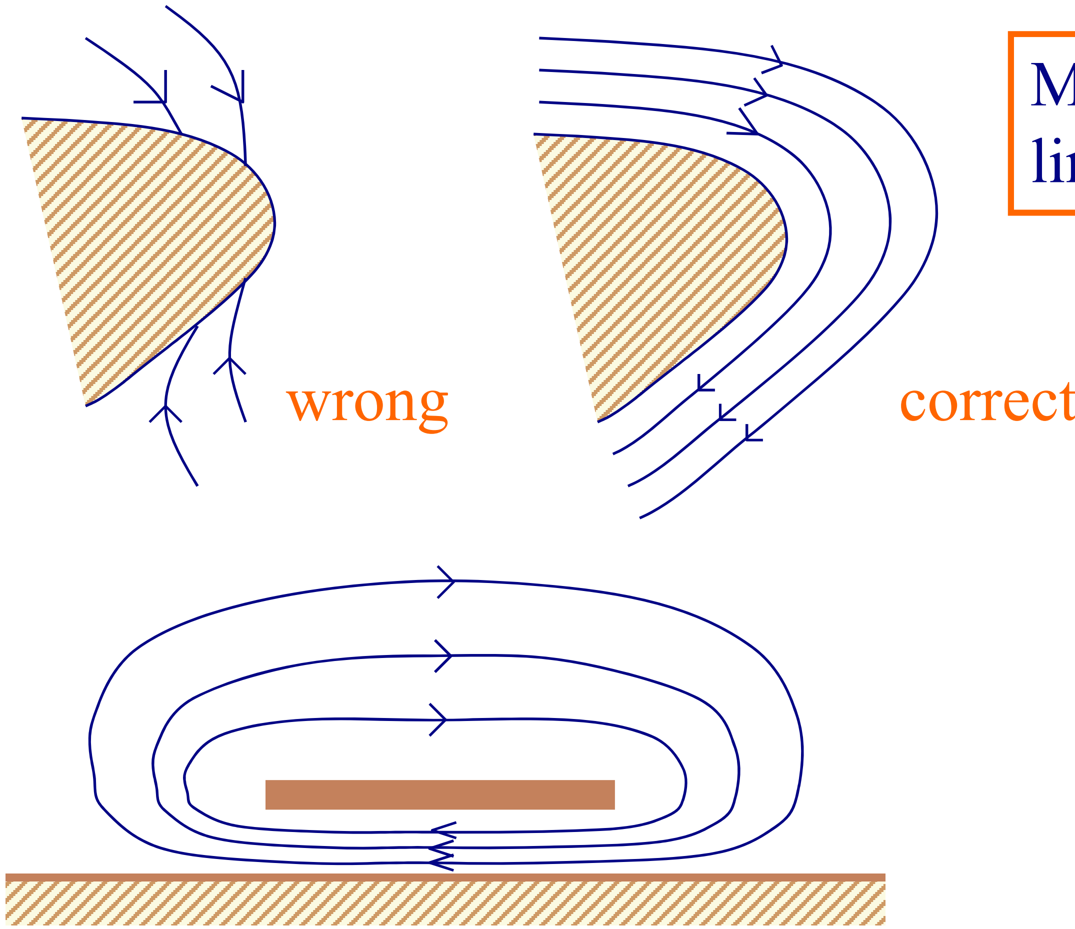
Boundary Conditions (6)

Electric field lines





Boundary Conditions (7)



Magnetic field lines

Energy and Power (1)

Consider a medium with ε, μ, σ

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \vec{M}_i = -\mu \frac{\partial \vec{H}}{\partial t} - \vec{M}_i \qquad (1)$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \vec{J}_i + \sigma \vec{E} \qquad (2)$$

$$\vec{H} \cdot (1) \quad \Rightarrow \quad \vec{H} \cdot \nabla \times \vec{E} = -\mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} - \vec{H} \cdot \vec{M}_i$$

$$\vec{E} \cdot (2) \quad \Rightarrow \quad \vec{E} \cdot \nabla \times \vec{H} = \varepsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \vec{E} \cdot \vec{J}_i + \sigma \vec{E} \cdot \vec{E}$$

$$\vec{H} \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \vec{H} = -\mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} - \vec{H} \cdot \vec{M}_i - \varepsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} - \vec{E} \cdot \vec{J}_i - \sigma \vec{E} \cdot \vec{E}$$

Energy and Power (2)

Vector identity: $\nabla \cdot \left(\vec{E} \times \vec{H}\right) = \vec{H} \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \vec{H}$

$$\nabla \cdot \left(\vec{E} \times \vec{H}\right) + \mu\vec{H} \cdot \frac{\partial \vec{H}}{\partial t} + \varepsilon\vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \sigma\vec{E} \cdot \vec{E} + \vec{H} \cdot \vec{M}_i + \vec{E} \cdot \vec{J}_i = 0$$

What is the physical meaning of this equation?

Use Gauss’ s theorem:

$$\oint\!\!\!\oint_S \left(\vec{E} \times \vec{H}\right) \cdot d\vec{s} + \iiint_V \left(\mu\vec{H} \cdot \frac{\partial \vec{H}}{\partial t} + \varepsilon\vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \sigma\vec{E} \cdot \vec{E} + \vec{H} \cdot \vec{M}_i + \vec{E} \cdot \vec{J}_i \right) dv = 0$$

Let’ s take a look at the unit of each term.

Energy and Power (3)

$$\oiint_S \left(\vec{E}\times\vec{H}\right)\cdot d\vec{s} + \iiint_V \left(\mu\vec{H}\cdot\frac{\partial\vec{H}}{\partial t} + \varepsilon\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} + \sigma\vec{E}\cdot\vec{E} + \vec{H}\cdot\vec{M}_i + \vec{E}\cdot\vec{J}_i \right) dv = 0$$

* $\vec{E}\times\vec{H}:\quad (\mathrm{V/m})\cdot(\mathrm{A/m}) = \mathrm{W/m^2}$

$$P_e = \oiint_S \left(\vec{E}\times\vec{H}\right)\cdot d\vec{s}$$ ⇨ Power passing through S

(leaving or entering?)

* $$\mu\vec{H}\cdot\frac{\partial\vec{H}}{\partial t} = \frac{1}{2}\mu\frac{\partial H^2}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right) = \frac{\partial w_m}{\partial t}$$

$w_m = \frac{1}{2}\mu H^2:\quad (\mathrm{H/m})\cdot(\mathrm{A/m})^2 = \mathrm{Joule/m^3}$ ⇨ Energy density

$W_m = \iiint_V w_m dv:$ energy associated with $\vec{H}$

Energy and Power (4)

$$\oint\!\!\oint_S \left(\vec{E}\times\vec{H}\right)\cdot d\vec{s} + \iiint_V \left(\mu\vec{H}\cdot\frac{\partial\vec{H}}{\partial t} + \varepsilon\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} + \sigma\vec{E}\cdot\vec{E} + \vec{H}\cdot\vec{M}_i + \vec{E}\cdot\vec{J}_i \right) dv = 0$$

* $$\varepsilon\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{1}{2}\varepsilon\frac{\partial E^2}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2\right) = \frac{\partial w_e}{\partial t}$$

$$w_e = \frac{1}{2}\varepsilon E^2 : \quad (\mathrm{F/m})\cdot(\mathrm{V/m})^2 = \mathrm{Joule/m^3}$$ ⇨ Energy density

$$W_e = \iiint_V w_e dv : \ \text{energy associated with } \vec{E}$$

Consider a special case (no loss & no source):

Power left

$$\oint\!\!\oint_S \left(\vec{E}\times\vec{H}\right)\cdot d\vec{s} = -\iiint_V \left(\mu\vec{H}\cdot\frac{\partial\vec{H}}{\partial t} + \varepsilon\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} \right) dv$$

Decreased power

Energy and Power (5)

$$\oint\!\!\oint_S \left(\vec{E}\times\vec{H}\right)\cdot d\vec{s} + \iiint_V \left(\mu\vec{H}\cdot\frac{\partial\vec{H}}{\partial t} + \varepsilon\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} + \sigma\vec{E}\cdot\vec{E} + \vec{H}\cdot\vec{M}_i + \vec{E}\cdot\vec{J}_i\right)dv = 0$$

* $\sigma\vec{E}\cdot\vec{E} = \sigma E^2 : \quad (\mathrm{S/m})\cdot(\mathrm{V/m})^2 = \mathrm{W/m^3}$ ⇨ Power density

$P_d = \iiint_V \sigma\vec{E}\cdot\vec{E}dv :$ power dissipated

* $P_S = -\iiint_V \left(\vec{H}\cdot\vec{M}_i + \vec{E}\cdot\vec{J}_i\right)dv$ ⇨ Power supplied by the sources

$$P_e + P_d - P_S + \frac{\partial}{\partial t}\left(W_m + W_e\right) = 0$$

Energy and Power (6)

$$P_S = P_e + P_d + \frac{\partial}{\partial t}\left(W_m + W_e\right)$$

Integral form

Conservation of Power

Denote

$$p_e = \nabla \cdot \left(\vec{E} \times \vec{H}\right), \qquad p_d = \sigma E^2$$

$$p_S = -\vec{H} \cdot \vec{M}_i - \vec{E} \cdot \vec{J}_i$$

$$p_S = p_e + p_d + \frac{\partial}{\partial t}\left(w_m + w_e\right)$$

Differential form

Conservation of Power

Time-Harmonic Fields (1)

Euler's formula: $e^{j\alpha} = \cos\alpha + j\sin\alpha$

$$\cos\alpha = \text{Re}(e^{j\alpha})$$

If $V(x, y, z, t)$ is oscillating at a single frequency:

$$\begin{aligned} V(x, y, z, t) &= V'(x, y, z)\cos(\omega t + \alpha) \\ &= V'(x, y, z)\,\text{Re}\left(e^{j(\omega t + \alpha)}\right) \\ &= \text{Re}\left[V'(x, y, z) e^{j\alpha} e^{j\omega t}\right] \\ &= \text{Re}\left[\hat{V}(x, y, z)\, e^{j\omega t}\right] \end{aligned}$$

$\hat{V}(x, y, z)$: phasor or complex quantity

Time-Harmonic Fields (2)

Extension to vectors:

$$E_{\substack{x\\y\\z}}(x,y,z,t) = \mathrm{Re}[\hat{E}_{\substack{x\\y\\z}}(x,y,z)e^{j\omega t}]$$

$$\left.\begin{aligned}\vec{E}(x,y,z,t) &= \mathrm{Re}[\hat{\vec{E}}(x,y,z)e^{j\omega t}] \\ \vec{H}(x,y,z,t) &= \mathrm{Re}[\hat{\vec{H}}(x,y,z)e^{j\omega t}]\end{aligned}\right\} \text{Time-harmonic field}$$

Same expressions for other quantities, such as **D, B, J, M**

Time-Harmonic Fields (4)

$$\nabla \times \hat{\vec{E}} = -j\omega \hat{\vec{B}} - \hat{\vec{M}}$$
$$\nabla \times \hat{\vec{H}} = j\omega \hat{\vec{D}} + \hat{\vec{J}}$$
$$\nabla \cdot \hat{\vec{J}} = -j\omega \hat{\rho}_e$$
$$\nabla \cdot \hat{\vec{M}} = -j\omega \hat{\rho}_m$$

Simplify notation

$$\nabla \times \vec{E} = -j\omega \vec{B} - \vec{M}$$
$$\nabla \times \vec{H} = j\omega \vec{D} + \vec{J}$$
$$\nabla \cdot \vec{D} = \rho_e$$
$$\nabla \cdot \vec{B} = \rho_m$$
$$\nabla \cdot \vec{J} = -j\omega \rho_e$$
$$\nabla \cdot \vec{M} = -j\omega \rho_m$$

Maxwell's equations for time-harmonic fields

Time-Harmonic Fields (5)

More general approach

Fourier transforms:

$$F(x,y,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(x,y,z,\omega) e^{j\omega t} d\omega$$

$$\hat{F}(x,y,z,\omega) = \int_{-\infty}^{\infty} F(x,y,z,t) e^{-j\omega t} dt$$

Expand

$$\vec{E}(x,y,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\vec{E}}(x,y,z,\omega) e^{j\omega t} d\omega$$

$$\vec{B}(x,y,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\vec{B}}(x,y,z,\omega) e^{j\omega t} d\omega$$

$$\vec{M}(x,y,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\vec{M}}(x,y,z,\omega) e^{j\omega t} d\omega$$

Complex Power (1)

Consider

$$\vec{S} = \vec{E} \times \vec{H} = \mathrm{Re}\left[\hat{\vec{E}} e^{j\omega t} \right] \times \mathrm{Re}\left[\hat{\vec{H}} e^{j\omega t} \right]$$

Poynting vector

$$= \frac{1}{2}\left[\hat{\vec{E}} e^{j\omega t} + \left(\hat{\vec{E}} e^{j\omega t} \right)^* \right] \times \frac{1}{2}\left[\hat{\vec{H}} e^{j\omega t} + \left(\hat{\vec{H}} e^{j\omega t} \right)^* \right]$$

$$= \frac{1}{4} \hat{\vec{E}} \times \hat{\vec{H}} e^{j2\omega t} + \frac{1}{4} \hat{\vec{E}} \times \hat{\vec{H}}^* + \frac{1}{4} \hat{\vec{E}}^* \times \hat{\vec{H}} + \frac{1}{4} \hat{\vec{E}}^* \times \hat{\vec{H}}^* e^{-j2\omega t}$$

$$= \frac{1}{2} \mathrm{Re}\left(\hat{\vec{E}} \times \hat{\vec{H}}^* \right) + \frac{1}{2} \mathrm{Re}\left(\hat{\vec{E}} \times \hat{\vec{H}} e^{j2\omega t} \right)$$

$$\vec{S}_{av} = \frac{1}{T_0} \int_0^{T_0} \vec{S} dt = \frac{1}{2} \mathrm{Re}\left(\hat{\vec{E}} \times \hat{\vec{H}}^* \right)$$

Complex Power (2)

$$\vec{S}_{av} = \frac{1}{2}\operatorname{Re}\left(\hat{\vec{E}} \times \hat{\vec{H}}^{*}\right)$$

Let $\hat{\vec{S}} = \frac{1}{2}\hat{\vec{E}} \times \hat{\vec{H}}^{*}$

$$\vec{S}_{av} = \operatorname{Re}(\hat{\vec{S}})$$

Simplify notation: $\vec{S} = \frac{1}{2}\vec{E} \times \vec{H}^{*}$

$$\vec{S}_{av} = \operatorname{Re}(\vec{S})$$

Complex Power (3)

What does the power conservation law look like?

$$\nabla \times \vec{E} = -j\omega\mu \vec{H} - \vec{M}_i$$

$$\nabla \times \vec{H} = j\omega\varepsilon \vec{E} + \sigma \vec{E} + \vec{J}_i$$

$$\vec{H}^* \cdot \nabla \times \vec{E} = -j\omega\mu \vec{H}^* \cdot \vec{H} - \vec{H}^* \cdot \vec{M}_i$$

$$\vec{E} \cdot \left(\nabla \times \vec{H}\right)^* = -j\omega\varepsilon \vec{E} \cdot \vec{E}^* + \sigma \vec{E} \cdot \vec{E}^* + \vec{E} \cdot \vec{J}_i^*$$

$$\nabla \cdot \left(\vec{E} \times \vec{H}^*\right) = \vec{H}^* \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \vec{H}^*$$

$$= -j\omega\mu \left|\vec{H}\right|^2 + j\omega\varepsilon \left|\vec{E}\right|^2 - \sigma \left|\vec{E}\right|^2 - \vec{H}^* \cdot \vec{M}_i - \vec{E} \cdot \vec{J}_i^*$$

Complex Power (4)

Examine a finite volume V enclosed by S:

$$\frac{1}{2}\oiint_S \left(\vec{E}\times\vec{H}^*\right)\cdot d\vec{s} + \frac{1}{2}\iiint_V \sigma\left|\vec{E}\right|^2 dv + j\frac{\omega}{2}\iiint_V \left(\mu\left|\vec{H}\right|^2 - \varepsilon\left|\vec{E}\right|^2\right) dv$$

$$= -\frac{1}{2}\iiint_V \left(\vec{E}\cdot\vec{J}_i^* + \vec{H}^*\cdot\vec{M}_i\right) dv$$

$$P_S = -\frac{1}{2}\iiint_V \left(\vec{E}\cdot\vec{J}_i^* + \vec{H}^*\cdot\vec{M}_i\right) dv$$ ⇨ Supplied complex power

$$P_e = \frac{1}{2}\oiint_S \left(\vec{E}\times\vec{H}^*\right)\cdot d\vec{s}$$ ⇨ exiting complex power

$$\bar{P}_d = \frac{1}{2}\iiint_V \sigma\left|\vec{E}\right|^2 dv$$ ⇨ Time-average dissipated power

Complex Power (5)

$$\overline{W}_m = \frac{1}{4}\iiint_V \mu\left|\vec{H}\right|^2 dv$$ ⇨ Time-average magnetic energy

$$\overline{W}_e = \frac{1}{4}\iiint_V \varepsilon\left|\vec{E}\right|^2 dv$$ ⇨ Time-average electric energy

$$\boxed{P_S = P_e + \overline{P}_d + j2\omega(\overline{W}_m - \overline{W}_e)}$$

Conservation of power

Define

$$p_s = -\tfrac{1}{2}(\vec{E}\cdot\vec{J}_i^{\,*} + \vec{H}^*\cdot\vec{M}_i)$$

$$p_e = \tfrac{1}{2}\nabla\cdot(\vec{E}\times\vec{H}^*)$$

$$\overline{p}_d = \tfrac{1}{2}\sigma\left|\vec{E}\right|^2$$

$$\overline{w}_m = \tfrac{1}{4}\mu\left|\vec{H}\right|^2$$

$$\overline{w}_e = \tfrac{1}{4}\varepsilon\left|\vec{E}\right|^2$$

Conservation of power

$$\boxed{p_S = p_e + \overline{p}_d + j2\omega(\overline{w}_m - \overline{w}_e)}$$

Complex Power (6)

$$P_S = P_e + \overline{P}_d + j2\omega(\overline{W}_m - \overline{W}_e)$$

Take the real part: $\mathrm{Re}(P_s) = \mathrm{Re}(P_e) + \overline{P}_d$

Take the imaginary part: $\mathrm{Im}(P_s) = \mathrm{Im}(P_e) + 2\omega(\overline{W}_m - \overline{W}_e)$

Law of conservation of power:

For arbitrary field: $P_s(t) = P_e(t) + P_d(t) + \frac{\partial}{\partial t}\left[W_m(t) + W_e(t)\right]$

For time-harmonic field: $\mathrm{Re}(P_s) = \mathrm{Re}(P_e) + \overline{P}_d$

Comparison suggests: $\frac{\overline{\partial W_m}}{\partial t} + \frac{\overline{\partial W_e}}{\partial t} = 0$

Complex Power (7)

Verification:

$$W_e(t) = \frac{1}{2}\iiint_V \varepsilon \vec{E}(t)\cdot\vec{E}(t)dv$$

$$= \frac{1}{2}\iiint_V \varepsilon E_0^2(t)\cdot\cos^2(\omega t + \theta_e)dv$$

$$= \frac{1}{4}\left[1+\cos(2\omega t + 2\theta_e)\right]\iiint_V \varepsilon E_0^2 dv$$

$$\overline{W_e} = \frac{1}{T_0}\int_0^{T_0} W_e(t)dt$$

$$= \frac{1}{4}\iiint_V \varepsilon E_0^2 dv = \text{constant}$$

⇨

$$\frac{\partial \overline{W}_e}{\partial t} = 0$$

Complex Power (8)

What's the meaning of this equation?

$$\mathrm{Im}(P_s) = \mathrm{Im}(P_e) + 2\omega(\overline{W}_m - \overline{W}_e)$$

Reactive power

Analogous to what happens in an LC circuit

Consider

$$\mathbf{J}_i = \mathbf{J}_{i0} e^{j\angle \mathbf{J}_i} \qquad \mathbf{E} = \mathbf{E}_0 e^{j\angle \mathbf{E}}$$

Complex supplied power density:

$$p_s = -\frac{1}{2}\mathbf{E}\cdot\mathbf{J}_i^* = -\frac{1}{2}\mathbf{E}_0\cdot\mathbf{J}_{i0} e^{j(\angle \mathbf{E} - \angle \mathbf{J}_i)}$$

Complex Power (9)

$$\mathrm{Re}(p_s) = -\frac{1}{2}\mathbf{E}_0 \cdot \mathbf{J}_{i0} \cos(\angle\mathbf{E} - \angle\mathbf{J}_i)$$

$$\mathrm{Im}(p_s) = -\frac{1}{2}\mathbf{E}_0 \cdot \mathbf{J}_{i0} \sin(\angle\mathbf{E} - \angle\mathbf{J}_i)$$

Instantaneous supplied power density:

$$p_s(t) = -\mathbf{E}(t) \cdot \mathbf{J}_i(t) = -\mathbf{E}_0 \cos(\omega t + \angle\mathbf{E}) \cdot \mathbf{J}_{i0} \cos(\omega t + \angle\mathbf{J}_i)$$

$$p_s(t) = -\frac{1}{2}\mathbf{E}_0 \cdot \mathbf{J}_{i0} \cos(\angle\mathbf{E} - \angle\mathbf{J}_i)\left[1 + \cos(2\omega t + 2\angle\mathbf{J}_i)\right]$$
$$+\frac{1}{2}\mathbf{E}_0 \cdot \mathbf{J}_{i0} \sin(\angle\mathbf{E} - \angle\mathbf{J}_i)\sin(2\omega t + 2\angle\mathbf{J}_i)$$

$\mathrm{Im}(p_s)$ represents the peak value of the reactive power density