ECE 222b
Applied Electromagnetics
SET 2

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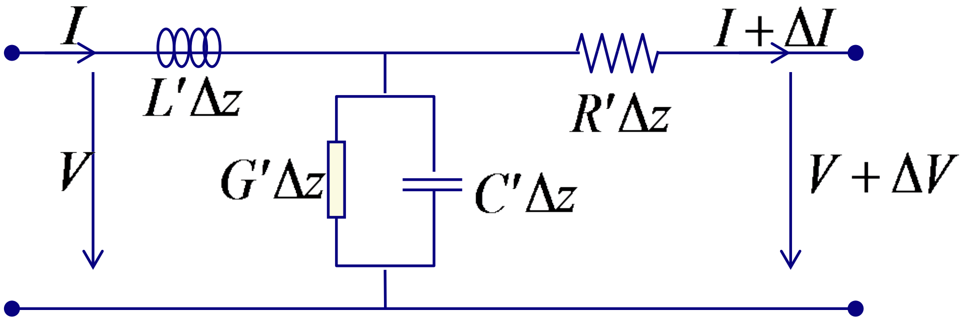
Review of Transmission Line (1)

Consider a segment of a transmission line:



Kirchhoff's laws:

$$V = j\omega L\Delta z I + R\Delta z(I+\Delta I) + V + \Delta V$$

$$\Rightarrow \quad \frac{dV}{dz} + (j\omega L + R)I = 0$$

$$I = G\Delta z V + j\omega C\Delta z V + I + \Delta I \quad \Rightarrow \quad \frac{dI}{dz} + (j\omega C + G)V = 0$$

Review of Transmission Line (2)

Try to eliminate *I*: $$\frac{d^2V}{dz^2} + (j\omega L + R)\frac{dI}{dz} = 0$$

$$\boxed{\frac{d^2V}{dz^2} - (j\omega L + R)(j\omega C + G)V = 0}$$

$$\boxed{\frac{d^2V}{dz^2} - \gamma^2 V = 0}$$

$$\gamma^2 = (j\omega L + R)(j\omega C + G)$$

Propagation constant

Similarly,

$$\boxed{\frac{d^2I}{dz^2} - \gamma^2 I = 0}$$

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Review of Transmission Line (3)

Consider a lossless line:

$$R = 0, \quad G = 0 \quad \Rightarrow \quad \gamma = j\omega\sqrt{LC}$$

Phase constant

$$\gamma = j\beta \qquad \beta = \omega\sqrt{LC}$$

$$\frac{d^2V}{dz^2} + \beta^2 V = 0 \quad \Rightarrow \quad V(z) = a_+ e^{-j\beta z} + a_- e^{j\beta z}$$

Consider $V_+(z) = a_+ e^{-j\beta z}$ Wave traveling toward +z direction!

$$V_+(z,t) = \mathrm{Re}\left[V_+(z) e^{-j\omega t}\right] = \mathrm{Re}\left(a_+ e^{j\omega t - j\beta z}\right)$$

$$= |a_+| \cos(\omega t - \beta z + \angle a_+)$$

Phase velocity

$$\omega t - \beta z_p = \text{constant} \quad \Rightarrow \quad \omega - \beta \frac{dz_p}{dt} = 0 \quad \Rightarrow \quad \frac{dz_p}{dt} = \frac{\omega}{\beta}$$

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Review of Transmission Line (4)

Consider a lossy line:

$$\gamma = \sqrt{(j\omega L + R)(j\omega C + G)} = \alpha + j\beta$$

$$\frac{d^2V}{dz^2} - \gamma^2 V = 0 \Longrightarrow V(z) = a_+ e^{-\gamma z} + a_- e^{\gamma z}$$

Consider $V_+(z) = a_+ e^{-\gamma z} = a_+ e^{-(\alpha + j\beta)z}$

$$V_+(z,t) = \mathrm{Re}\left[V_+(z) e^{j\omega t}\right] = \mathrm{Re}\left(a_+ e^{j\omega t - j\beta z - \alpha z}\right)$$

$$= |a_+| e^{-\alpha z} \cos(\omega t - \beta z + \angle a_+)$$

α : attenuation constant β : phase constant

Review of Transmission Line (5)

Consider the general case:

$$V(z) = a_+ e^{-\gamma z} + a_- e^{\gamma z}$$

Define

$$\Gamma = \frac{V_-(0)}{V_+(0)} = \frac{a_-}{a_+} \Rightarrow V(z) = a_+ e^{-\gamma z} + a_+ \Gamma e^{\gamma z}$$

Reflection coefficient:

$$\Gamma(z) = \frac{V_-(z)}{V_+(z)} = \frac{a_- e^{\gamma z}}{a_+ e^{-\gamma z}} = \Gamma e^{2\gamma z}$$

$$\Gamma(z_1) = \Gamma e^{2\gamma z_1}$$

$$\Gamma(z_2) = \Gamma e^{2\gamma z_2}$$

$$\Rightarrow \Gamma(z_1) = \Gamma(z_2) e^{2\gamma (z_1 - z_2)}$$

Review of Transmission Line (6)

Find the current:

$$\frac{dV}{dz} = -\gamma a_+ e^{-\gamma z} + \gamma a_+ \Gamma e^{-\gamma z} = -(j\omega L + R)I$$

$$I(z) = \frac{\gamma}{j\omega L + R} a_+ e^{-\gamma z} - \frac{\gamma}{j\omega L + R} a_+ \Gamma e^{\gamma z}$$

$$= \frac{\gamma}{j\omega L + R} a_+ (e^{-\gamma z} - \Gamma e^{\gamma z})$$

Define the characteristic impedance:

$$Z_0 = \frac{V_+}{I_+} = \frac{j\omega L + R}{\gamma} = \sqrt{\frac{j\omega L + R}{j\omega C + G}}$$

Review of Transmission Line (7)

Impedance at z:

$$Z(z)=\frac{V(z)}{I(z)}=\frac{a_+e^{-\gamma z}+a_+\Gamma e^{\gamma z}}{\frac{1}{Z_0}a_+(e^{-\gamma z}-\Gamma e^{\gamma z})}=Z_0\frac{1+\Gamma e^{2\gamma z}}{1-\Gamma e^{2\gamma z}}=Z_0\frac{1+\Gamma(z)}{1-\Gamma(z)}$$

$$\boxed{Z(z)=Z_0\frac{1+\Gamma(z)}{1-\Gamma(z)}} \qquad \boxed{\Gamma(z)=\frac{Z(z)-Z_0}{Z(z)+Z_0}}$$

Observation:
1. Z_0 is independent of z
2. $Z(z)$ depends on z
3. $\Gamma(z)$ depends on z
4. For a lossless line, $\gamma = j\beta$

$$|\Gamma(z)|=\left|\Gamma e^{2j\beta z}\right|=|\Gamma| \text{ is invariant along } z$$

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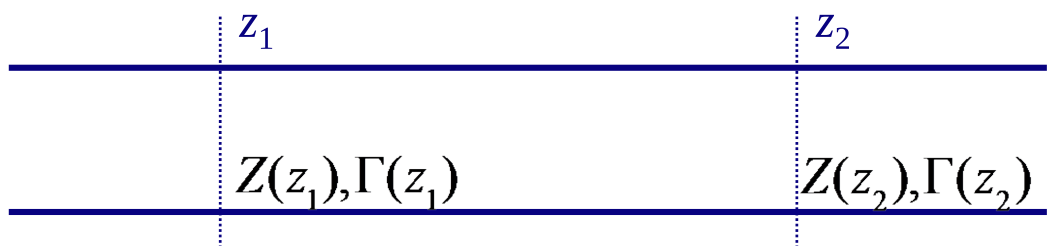
Review of Transmission Line (8)

Typical problem:

Given information at z_1, find information at z_2



$$\Gamma(z) = \Gamma e^{2\gamma z} \quad \Rightarrow \quad \Gamma(z_2) = \Gamma(z_1) e^{2\gamma(z_2 - z_1)}$$

$$Z(z_2) = Z_0 \frac{1 + \Gamma(z_2)}{1 - \Gamma(z_2)} = Z_0 \frac{1 + \Gamma(z_1) e^{2\gamma(z_2 - z_1)}}{1 - \Gamma(z_1) e^{2\gamma(z_2 - z_1)}}$$

$$Z(z_1) = Z_0 \frac{1 + \Gamma(z_1)}{1 - \Gamma(z_1)} \quad \Rightarrow \quad \Gamma(z_1) = \frac{Z(z_1) - Z_0}{Z(z_1) + Z_0}$$

Review of Transmission Line (9)

$$Z(z_2) = Z_0 \frac{1 + \dfrac{Z(z_1) - Z_0}{Z(z_1) + Z_0} e^{2\gamma(z_2 - z_1)}}{1 - \dfrac{Z(z_1) - Z_0}{Z(z_1) + Z_0} e^{2\gamma(z_2 - z_1)}} = Z_0 \frac{Z_1 + Z_0 + (Z_1 - Z_0) e^{2\gamma(z_2 - z_1)}}{Z_1 + Z_0 - (Z_1 - Z_0) e^{2\gamma(z_2 - z_1)}}$$

$$Z(z_2) = Z_0 \frac{Z_1(e^{\gamma l} + e^{-\gamma l}) - Z_0(e^{\gamma l} - e^{-\gamma l})}{Z_1(e^{-\gamma l} - e^{\gamma l}) + Z_0(e^{\gamma l} + e^{-\gamma l})} \qquad l = z_2 - z_1$$

$$= Z_0 \frac{Z_1 \cosh \gamma l - Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l - Z_1 \sinh \gamma l}$$

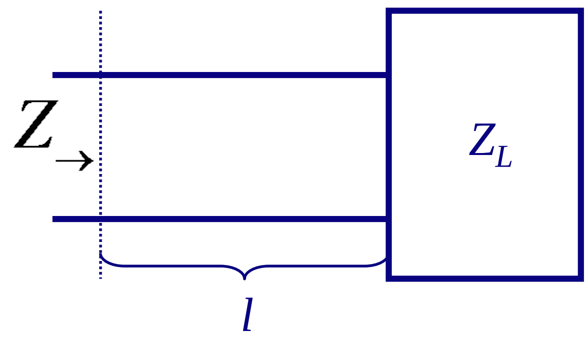
For a lossless line: $\gamma = j\beta$

$$Z_2 = Z_0 \frac{Z_1 \cos \beta l - jZ_0 \sin \beta l}{Z_0 \cos \beta l - jZ_1 \sin \beta l}$$

$$Z_1 = Z_0 \frac{Z_2 \cos \beta l + jZ_0 \sin \beta l}{Z_0 \cos \beta l + jZ_2 \sin \beta l}$$

Review of Transmission Line (10)

$$Z_{\rightarrow} = Z_0 \frac{Z_L \cos\beta l + jZ_0 \sin\beta l}{Z_0 \cos\beta l + jZ_L \sin\beta l}$$



If $Z_L = 0, \quad Z_{\rightarrow} = jZ_0 \tan\beta l$

If $Z_L = \infty, \quad Z_{\rightarrow} = -jZ_0 \cot\beta l$

$Z_{\rightarrow}$ = input impedance