ECE 222b
Applied Electromagnetics
Notes Set 3a

Instructor: Prof. Vitaliy Lomakin

Department of Electrical and Computer Engineering
University of California, San Diego

Uniform Plane Waves (1)

Consider Maxwell's equations:

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} - \vec{M}_i$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} + \sigma\vec{E} + \vec{J}_i$$

In a lossless medium, ε and μ are real and σ = 0:

$$\begin{aligned}\nabla \times \nabla \times \vec{E} &= -j\omega\mu\nabla \times \vec{H} - \nabla \times \vec{M}_i \\ &= -j\omega\mu(j\omega\varepsilon\vec{E} + \vec{J}_i) - \nabla \times \vec{M}_i \\ &= \omega^2\mu\varepsilon\vec{E} - j\omega\mu\vec{J}_i - \nabla \times \vec{M}_i\end{aligned}$$

Since $\nabla \times \nabla \times \vec{E} = \nabla\nabla \cdot \vec{E} - \nabla^2\vec{E} = \nabla(\frac{1}{\varepsilon}\rho_e) - \nabla^2\vec{E}$

$$\boxed{\nabla^2\vec{E} + \omega^2\mu\varepsilon\vec{E} = j\omega\mu\vec{J}_i + \nabla \times \vec{M}_i + \nabla(\tfrac{1}{\varepsilon}\rho_e)}$$

Uniform Plane Waves (2)

Similarly

$$\nabla^2 \vec{H} + \omega^2 \mu\varepsilon \vec{H} = j\omega\varepsilon \vec{M}_i - \nabla \times \vec{J}_i + \nabla(\tfrac{1}{\mu}\rho_m)$$

In source-free region: $\vec{J}_i = \vec{M}_i = \rho_e = \rho_m = 0$

$$\nabla^2 \vec{E} + \omega^2 \mu\varepsilon \vec{E} = 0$$
$$\nabla^2 \vec{H} + \omega^2 \mu\varepsilon \vec{H} = 0$$

or

$$\nabla^2 \vec{E} - \gamma^2 \vec{E} = 0$$
$$\nabla^2 \vec{H} - \gamma^2 \vec{H} = 0$$

Wave equations

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Laplacian operator

$$\gamma = j\beta = j\omega\sqrt{\mu\varepsilon}$$

Propagation constant

Uniform Plane Waves (3)

Special case: $\vec{E} = \hat{x} E_x(z)$

Wave equation $\Rightarrow$ $$\frac{\partial^2 E_x}{\partial z^2} - \gamma^2 E_x = 0$$

Two solutions: $Ae^{-\gamma z}, \qquad Be^{\gamma z}$

Consider $E_x^+ = Ae^{-\gamma z}$

$$\nabla \times \vec{E} = -j\omega\mu \vec{H} \qquad \Rightarrow \qquad H_y^+ = \frac{\gamma}{j\omega\mu} Ae^{-\gamma z} = \sqrt{\frac{\varepsilon}{\mu}} Ae^{-\gamma z}$$

Define intrinsic impedance:

$$\eta = \sqrt{\mu / \varepsilon}$$ then $$H_y^+ = \frac{1}{\eta} Ae^{-\gamma z}$$

Uniform Plane Waves (4)

Wave impedance: $$Z_w = \frac{E_x^+}{H_y^+} = \sqrt{\frac{\mu}{\varepsilon}} = \eta$$

In time domain:

$$E_x^+(z,t) = \mathrm{Re}\left[E_x^+ e^{j\omega t}\right] = \mathrm{Re}\left[A e^{j(\omega t - \beta z)}\right] = A\cos(\omega t - \beta z)$$

Uniform Plane Wave

Equi-phase plane: $\omega t - \beta z_p = \text{constant}$

$$\omega - \beta \frac{dz_p}{dt} = 0 \quad \Rightarrow$$

Phase velocity: $$v_p = \frac{dz_p}{dt} = \frac{\omega}{\beta}$$

Here, $\beta = \omega\sqrt{\mu\varepsilon}, \quad v_p = \dfrac{1}{\sqrt{\mu\varepsilon}}$

Speed of light

In vacuum, $\mu = \mu_0, \ \varepsilon = \varepsilon_0, \ v_p = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3\times 10^8 \, \mathrm{m/s}$

Uniform Plane Waves (5)

Wavelength: $\lambda = T \cdot v_p = \frac{1}{f} v_p = \frac{2\pi}{\omega} v_p = \frac{2\pi}{\beta} \Rightarrow \beta = \frac{2\pi}{\lambda}$

wavenumber

Consider

$$E_x^+(z,t) = A\cos(\omega t - \beta z)$$

$$H_y^+(z,t) = \frac{A}{\eta}\cos(\omega t - \beta z)$$

} Fields

Energy density

$$w_e = \frac{1}{2}\varepsilon|E|^2 = \frac{\varepsilon}{2}A^2\cos^2(\omega t - \beta z)$$

$$w_m = \frac{1}{2}\mu|H|^2 = \frac{\mu}{2}\frac{A^2}{\eta^2}\cos^2(\omega t - \beta z) = \frac{\varepsilon}{2}A^2\cos^2(\omega t - \beta z)$$

$$\vec{S}(z,t) = \vec{E} \times \vec{H} = \hat{z}\frac{A^2}{\eta}\cos^2(\omega t - \beta z)$$

} Power flow density

Uniform Plane Waves (6)

Energy velocity: $$v_e = \frac{\text{power flow density}}{\text{energy density}} = \frac{S}{w_e + w_m} = \frac{1}{\sqrt{\mu\varepsilon}}$$

Consider

$$\begin{aligned}\vec{E} &= \hat{x}E_0 \cos\left[(\omega + \Delta\omega)t - (\beta + \Delta\beta)z\right] \\ &\quad + \hat{x}E_0 \cos\left[(\omega - \Delta\omega)t - (\beta - \Delta\beta)z\right] \\ &= \hat{x}2E_0 \cos(\Delta\omega t - \Delta\beta z)\cos(\omega t - \beta z)\end{aligned}$$

$$\Delta\omega t - \Delta\beta z = \text{constant}$$ ⟸ Wave packet

Group velocity: $$v_g = \frac{dz}{dt} = \frac{\Delta\omega}{\Delta\beta} = \frac{1}{\frac{\Delta\beta}{\Delta\omega}} = \frac{1}{\frac{d\beta}{d\omega}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

Uniform Plane Waves (7)

Phase velocity: $v_p = \dfrac{\omega}{\beta}$

$$\frac{d\beta}{d\omega} = \frac{d}{d\omega}\left(\frac{\omega}{v_p}\right) = \frac{1}{v_p} - \frac{\omega}{v_p^{\,2}}\frac{dv_p}{d\omega} \quad \Rightarrow \quad \boxed{v_g = \frac{v_p}{1 - \dfrac{\omega}{v_p}\dfrac{dv_p}{d\omega}}}$$

(1) No dispersion: $dv_p/d\omega = 0 \quad \Rightarrow \quad v_g = v_p$

(2) Normal dispersion: $dv_p/d\omega < 0 \quad \Rightarrow \quad v_g < v_p$

(3) Anomalous dispersion: $dv_p/d\omega > 0 \quad \Rightarrow \quad v_g > v_p$

Uniform Plane Waves (8)

Standing wave:

In frequency domain

$$\begin{cases} E_x(z) = Ae^{-j\beta z} + Ae^{j\beta z} = 2A\cos\beta z \\ H_y(z) = \dfrac{2A}{j\eta}\sin\beta z \end{cases}$$

In time domain

$$\begin{cases} E_x(z,t) = \mathrm{Re}[E_x e^{j\omega t}] = 2A\cos\beta z\cos\omega t \\ H_y(z,t) = \dfrac{2A}{\eta}\sin\beta z\sin\omega t \end{cases}$$

1. The phase is independent of z, or $v_p = 0$

2. $w_e = \dfrac{1}{2}\varepsilon|E|^2 = 2\varepsilon A^2\cos^2\beta z\cos^2\omega t$

$$w_m = \frac{1}{2}\mu|H|^2 = 2\varepsilon A^2\sin^2\beta z\sin^2\omega t$$

Uniform Plane Waves (9)

$$\vec{S}(z,t) = \vec{E} \times \vec{H} = \hat{z}\frac{4A^2}{\eta}\cos\beta z\cos\omega t\sin\beta z\sin\omega t$$

$$= \hat{z}\frac{A^2}{\eta}\sin(2\beta z)\sin(2\omega t)$$

➡ Time average = 0

$$\hat{\vec{S}} = \frac{1}{2}\vec{E} \times \vec{H}^* = \hat{z}\frac{2jA^2}{\eta}\sin\beta z\cos\beta z = \hat{z}\frac{jA^2}{\eta}\sin(2\beta z)$$

$$\text{Time-average power flow} = \mathrm{Re}(\hat{\vec{S}}) = 0$$

For a more general case:

$$E_x = Ae^{-j\beta z} + Be^{j\beta z} = A(\cos\beta z - j\sin\beta z) + B(\cos\beta z + j\sin\beta z)$$

$$= (A+B)\cos\beta z - j(A-B)\sin\beta z$$

$$|E_x| = \sqrt{(A+B)^2\cos^2\beta z - (A-B)^2\sin^2\beta z} = \sqrt{A^2 + B^2 + 2AB\cos(2\beta z)}$$

Uniform Plane Waves (10)

$$\left|E_x\right|_{\max} = |A| + |B| \qquad \left|E_x\right|_{\min} = |A| - |B|$$

$$\angle E_x = \tan^{-1}\left[-\tfrac{A-B}{A+B}\tan\beta z\right]$$

Standing wave ratio (SWR):

$$\mathrm{SWR} = \frac{\left|E_x\right|_{\max}}{\left|E_x\right|_{\min}} = \frac{|A| + |B|}{|A| - |B|} = \frac{1 + \frac{|B|}{|A|}}{1 - \frac{|B|}{|A|}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

For a pure traveling wave: $B = 0, \quad \mathrm{SWR} = 1$

For a pure standing wave: $B = A, \quad \mathrm{SWR} = \infty$

For a general wave: $1 \le \mathrm{SWR} < \infty$

Uniform Plane Waves (11)

Uniform plane wave in a lossy medium:

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} - \vec{M}_i$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} + \sigma\vec{E} + \vec{J}_i = (j\omega\varepsilon + \sigma)\vec{E} + \vec{J}_i$$

In a source-free region:

$$\nabla^2\vec{E} - \gamma^2\vec{E} = 0 \qquad \gamma^2 = j\omega\mu(j\omega\varepsilon + \sigma)$$

$$Z_w = \frac{j\omega\mu}{\gamma} = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon + \sigma}}$$

A simple case: $\vec{E} = \hat{x}E_x(z)$

$$\frac{d^2E_x}{dz^2} - \gamma^2 E_x = 0 \qquad \gamma = \alpha + j\beta$$

Uniform Plane Waves (12)

Possible solutions: $E_x^+ = Ae^{-\gamma z} = Ae^{-\alpha z}e^{-j\beta z}$

$$E_x^- = Be^{\gamma z} = Be^{\alpha z}e^{j\beta z}$$

How to find α and β?

$$\gamma = \sqrt{j\omega\mu(j\omega\varepsilon + \sigma)} = \alpha + j\beta$$

⇦ Complex equation

$$-\omega^2\mu\varepsilon = \alpha^2 - \beta^2, \qquad \omega\mu\sigma = 2\alpha\beta$$

$$\beta = \frac{\omega\mu\sigma}{2\alpha}, \qquad \alpha^2 - \frac{(\omega\mu\sigma)^2}{4\alpha^2} = -\omega^2\mu\varepsilon$$

$$\alpha^4 + \omega^2\mu\varepsilon\alpha^2 - \frac{1}{4}(\omega\mu\sigma)^2 = 0$$

Uniform Plane Waves (13)

$$\alpha = \pm\omega\sqrt{\mu\varepsilon}\sqrt{\tfrac{1}{2}\left(\sqrt{1+\left(\tfrac{\sigma}{\omega\varepsilon}\right)^2}-1\right)} \Rightarrow \alpha = \pm|\alpha|$$

$$\beta = \pm\omega\sqrt{\mu\varepsilon}\sqrt{\tfrac{1}{2}\left(\sqrt{1+\left(\tfrac{\sigma}{\omega\varepsilon}\right)^2}+1\right)} \Rightarrow \beta = \pm|\beta|$$

Four solutions: $\gamma = \alpha + j\beta$

$$\gamma_1 = |\alpha| + j|\beta|$$ ← Desired solution

$$\gamma_2 = -|\alpha| + j|\beta|$$

$$\gamma_3 = |\alpha| - j|\beta|$$

$$\gamma_4 = -|\alpha| - j|\beta|$$

Uniform Plane Waves (14)

$$\alpha = \pm\omega\sqrt{\mu\varepsilon}\sqrt{\tfrac{1}{2}\left(\sqrt{1+\left(\tfrac{\sigma}{\omega\varepsilon}\right)^2}-1\right)}$$

$$\beta = \pm\omega\sqrt{\mu\varepsilon}\sqrt{\tfrac{1}{2}\left(\sqrt{1+\left(\tfrac{\sigma}{\omega\varepsilon}\right)^2}+1\right)}$$

$$\eta = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon+\sigma}}$$

1. For perfect conductor: $\sigma = \infty$

$$\alpha \to \infty, \qquad \beta \to \infty, \qquad \eta \to 0$$

2. For perfect dielectric: $\sigma = 0$

$$\alpha = 0, \qquad \beta = \omega\sqrt{\mu\varepsilon}, \qquad \eta = \sqrt{\tfrac{\mu}{\varepsilon}}$$

Uniform Plane Waves (15)

3. For good dielectric: $\left(\frac{\sigma}{\omega\varepsilon}\right)^2 << 1$

$$\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} = 1+\frac{1}{2}\left(\frac{\sigma}{\omega\varepsilon}\right)^2 - \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^4 + \cdots \approx 1+\frac{1}{2}\left(\frac{\sigma}{\omega\varepsilon}\right)^2$$

$$\alpha \approx \omega\sqrt{\mu\varepsilon}\cdot\frac{1}{2}\left(\frac{\sigma}{\omega\varepsilon}\right) = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}, \quad \beta \approx \omega\sqrt{\mu\varepsilon}, \quad \eta \approx \sqrt{\frac{\mu}{\varepsilon}}$$

4. For good conductor: $\frac{\sigma}{\omega\varepsilon} >> 1$

$$\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} = \frac{\sigma}{\omega\varepsilon} + \frac{1}{2}\left(\frac{\omega\varepsilon}{\sigma}\right) - \frac{1}{8}\left(\frac{\omega\varepsilon}{\sigma}\right)^3 + \cdots$$

$$\alpha \approx \omega\sqrt{\mu\varepsilon}\cdot\sqrt{\frac{\sigma}{2\omega\varepsilon}} = \sqrt{\frac{\omega\mu\sigma}{2}}, \quad \beta \approx \sqrt{\frac{\omega\mu\sigma}{2}}, \quad \eta = \sqrt{\frac{\omega\mu}{2\sigma}}(1+j)$$

Uniform Plane Waves (16)

Skin depth: The field amplitude reduces to $e^{-1} \approx 36.8\%$

$$E_x^+ = Ae^{-\gamma z} = Ae^{-\alpha z}e^{-j\beta z}$$

$$\delta = \frac{1}{\alpha} = \frac{1}{\omega\sqrt{\mu\varepsilon}\sqrt{\frac{1}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right)}} \quad \text{(m)}$$

1. For perfect conductor: $\delta \to 0$

2. For perfect dielectric: $\delta \to \infty$

3. For good conductor: $\delta \approx \sqrt{\frac{2}{\omega\mu\sigma}}$

4. For good dielectric: $\delta \approx \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$

Polarization (1)

Polarization: The direction of the electric field

1. $E_x = Ae^{-j\beta z}, \qquad H_y = \dfrac{A}{\eta} e^{-j\beta z}$

→ Linearly polarized in the x direction.

2. $E_y = Be^{-j\beta z}, \qquad H_x = -\dfrac{B}{\eta} e^{-j\beta z}$

→ Linearly polarized in the y direction.

3. Consider their combination:

$$\vec{E} = \hat{x}Ae^{-j\beta z} + \hat{y}Be^{-j\beta z} = (\hat{x}A + \hat{y}B)e^{-j\beta z}$$

$$\vec{H} = \hat{y}\tfrac{A}{\eta}e^{-j\beta z} - \hat{x}\tfrac{B}{\eta}e^{-j\beta z} = (\hat{y}A - \hat{x}B)\tfrac{1}{\eta}e^{-j\beta z}$$

Polarization (2)

In time domain:

$$E_x(z,t) = \mathrm{Re}(E_x e^{j\omega t}) = \mathrm{Re}\left(A e^{j(\omega t - \beta z)}\right)$$
$$= \mathrm{Re}\left(|A| e^{j(\omega t - \beta z + a)}\right) = |A| \cos(\omega t - \beta z + a)$$

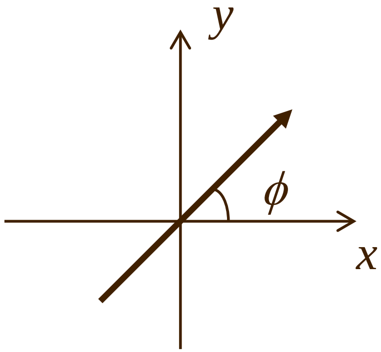
$$E_y(z,t) = |B| \cos(\omega t - \beta z + b)$$

$$\vec{E}(z,t) = \hat{x}|A|\cos(\omega t - \beta z + a) + \hat{y}|B|\cos(\omega t - \beta z + b)$$

1. If $a = b$ then $\vec{E}(z,t) = \left(\hat{x}|A| + \hat{y}|B|\right)\cos(\omega t - \beta z + a)$

→ Linearly polarized in the direction:

$$\phi = \tan^{-1}\left(\frac{|B|}{|A|}\right)$$



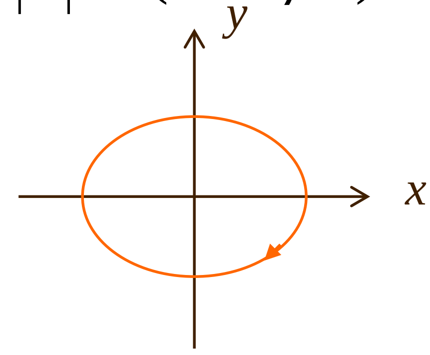
Polarization (3)

2. If $a = 0, b = \frac{\pi}{2}$ then

$$\vec{E}(z,t) = \hat{x}|A|\cos(\omega t - \beta z) - \hat{y}|B|\sin(\omega t - \beta z)$$

$$E_x(z,t) = |A|\cos(\omega t - \beta z), \qquad E_y(z,t) = -|B|\sin(\omega t - \beta z)$$

$$\left(\frac{E_x}{|A|}\right)^2 + \left(\frac{E_y}{|B|}\right)^2 = 1$$



⇒ elliptically polarized { Left-hand; Counter-clockwise

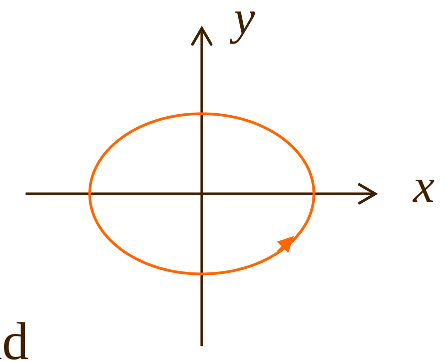
If $|A| = |B| \quad \Rightarrow$ circularly polarized

Polarization (4)

3. If $a = \frac{\pi}{2}, b = 0$ then

$$\vec{E}(z,t) = -\hat{x}|A|\sin(\omega t - \beta z) + \hat{y}|B|\cos(\omega t - \beta z)$$

$$\left(\frac{E_x}{|A|}\right)^2 + \left(\frac{E_y}{|B|}\right)^2 = 1$$



⇒ elliptically polarized { Right-hand, Clock wise

If $|A| = |B| \Rightarrow$ circularly polarized

Polarization (5)

Summary

1. Linearly polarized:

$$\vec{E}(z) = \left(\hat{x}|A| \pm \hat{y}|B|\right)e^{-j\beta z}$$

2. Circularly polarized:

$$\vec{E}(z) = \left(\hat{x} - j\hat{y}\right)E_0 e^{-j\beta z}$$ right-hand

$$\vec{E}(z) = \left(\hat{x} + j\hat{y}\right)E_0 e^{-j\beta z}$$ left-hand

3. Elliptically polarized:

$$\vec{E}(z) = \left(\hat{x}|A| - j\hat{y}|B|\right)e^{-j\beta z}$$ right-hand

$$\vec{E}(z) = \left(\hat{x}|A| + j\hat{y}|B|\right)e^{-j\beta z}$$ left-hand

Polarization (6)

Observation #1:

An elliptically (or circularly) polarized wave can be decomposed into two linearly polarized waves.

Observation #2:

A linearly polarized wave can be decomposed into two elliptically (or circularly) polarized waves.

$$\vec{E} = \hat{x}|A|e^{-j\beta z} = \left(\hat{x}\frac{|A|}{2} + j\hat{y}\frac{|B|}{2} + \hat{x}\frac{|A|}{2} - j\hat{y}\frac{|B|}{2} \right) e^{-j\beta z}$$

$$= \left(\hat{x}\frac{|A|}{2} + j\hat{y}\frac{|B|}{2} \right) e^{-j\beta z} \quad \leftarrow \quad \text{left - hand}$$

$$+ \left(\hat{x}\frac{|A|}{2} - j\hat{y}\frac{|B|}{2} \right) e^{-j\beta z} \quad \leftarrow \quad \text{right - hand}$$

Polarization (7)

Observation #3:

For a linearly polarized wave:

$$\vec{S}(t) = \vec{E} \times \vec{H} = \hat{z}\frac{|A|^2}{\eta}\cos^2(\omega t - \beta z)$$

For a circularly polarized wave:

$$\vec{E} = (\hat{x} \pm j\hat{y})Ae^{-j\beta z}, \qquad \vec{H} = (\mp\hat{x} - j\hat{y})j\frac{A}{\eta}e^{-j\beta z}$$

$$\vec{E}(t) = \hat{x}A\cos(\omega t - \beta z) \pm \hat{y}A\sin(\omega t - \beta z)$$

$$\vec{H}(t) = \mp\hat{x}\frac{A}{\eta}\sin(\omega t - \beta z) + \hat{y}\frac{A}{\eta}\cos(\omega t - \beta z)$$

Steady power flow

$$\vec{S}(t) = \vec{E} \times \vec{H} = \hat{z}\frac{|A|^2}{\eta}\cos^2(\omega t - \beta z) + \hat{z}\frac{|A|^2}{\eta}\sin^2(\omega t - \beta z) = \hat{z}\frac{|A|^2}{\eta}$$

Uniform Plane Waves (1)

Wave equation:

Vector form $\nabla^2\vec{E}-\gamma^2\vec{E}=0 \qquad \gamma^2=-\omega^2\mu\varepsilon$

Scalar form $\nabla^2E_x-\gamma^2E_x=0 \quad \nabla^2E_y-\gamma^2E_y=0 \quad \nabla^2E_z-\gamma^2E_z=0$

In rectangular coordinates: $\dfrac{\partial^2E_x}{\partial x^2}+\dfrac{\partial^2E_x}{\partial y^2}+\dfrac{\partial^2E_x}{\partial z^2}-\gamma^2E_x=0$

Try separation of variables: $E_x=X(x)Y(y)Z(z)$

$$YZ\frac{\partial^2X}{\partial x^2}+XZ\frac{\partial^2Y}{\partial y^2}+XY\frac{\partial^2Z}{\partial z^2}-\gamma^2XYZ=0$$

$$\frac{1}{X}\frac{\partial^2X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2Y}{\partial y^2}+\frac{1}{Z}\frac{\partial^2Z}{\partial z^2}-\gamma^2=0$$

Uniform Plane Waves (2)

Separation: $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \gamma_x{}^2$ $\quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = \gamma_y{}^2$ $\quad \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = \gamma_z{}^2$

$$\frac{\partial^2 X}{\partial x^2} - \gamma_x{}^2 X = 0 \quad \Rightarrow \quad A_x e^{\pm \gamma_x x} \qquad \gamma_x{}^2 + \gamma_y{}^2 + \gamma_z{}^2 = \gamma^2$$

$$\frac{\partial^2 Y}{\partial y^2} - \gamma_y{}^2 Y = 0 \quad \Rightarrow \quad A_y e^{\pm \gamma_y y}$$

$$\frac{\partial^2 Z}{\partial z^2} - \gamma_z{}^2 Z = 0 \quad \Rightarrow \quad A_z e^{\pm \gamma_z z}$$

$$E_x = XYZ = A_x e^{\pm \gamma_x x} \cdot A_y e^{\pm \gamma_y y} \cdot A_z e^{\pm \gamma_z z}$$

$$= A_x A_y A_z e^{\pm(\gamma_x x + \gamma_y y + \gamma_z z)} = A e^{\pm \vec{\gamma} \cdot \vec{r}}$$

$$\vec{\gamma} = \gamma_x \hat{x} + \gamma_y \hat{y} + \gamma_z \hat{z}$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

Uniform Plane Waves (3)

Similarly $E_y = Be^{\pm\vec{\gamma}\cdot\vec{r}}$ $E_z = Ce^{\pm\vec{\gamma}\cdot\vec{r}}$

$$\vec{E} = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z = \vec{E}_0 e^{\pm\vec{\gamma}\cdot\vec{r}} = \vec{E}_0 e^{-(\vec{\alpha}+j\vec{\beta})\cdot\vec{r}} \qquad \gamma = \alpha + j\beta$$

$$\vec{E}(\vec{r},t) = \mathrm{Re}[\vec{E}_0 e^{-(\vec{\alpha}+j\vec{\beta})\cdot\vec{r}} e^{j\omega t}] = \left|\vec{E}_0\right| e^{-\vec{\alpha}\cdot\vec{r}} \cos(\omega t - \vec{\beta}\cdot\vec{r} + \angle\vec{E}_0)$$

Equi-amplitude: $\vec{\alpha}\cdot\vec{r} = \text{constant}$

Equi-phase: $\vec{\beta}\cdot\vec{r} = \text{constant}$

} **Uniform Plane Waves**

Equi-phase plane: $\omega t - \vec{\beta}\cdot\vec{r} + \angle\vec{E}_0 = \text{constant}$

$$\omega\Delta t - \vec{\beta}\cdot\Delta\vec{r} = 0$$

$$\vec{v}_p = \frac{\Delta\vec{r}}{\Delta t} = \hat{\beta}\frac{\omega}{\beta}$$

⇦ Phase velocity

Uniform Plane Waves (4)

Useful mathematical formulas:

For $e^{\alpha x}$, $\quad \frac{d}{dx} e^{\alpha x} = \alpha e^{\alpha x}$

For $e^{\vec{\gamma}\cdot\vec{r}}$, $\quad \nabla e^{\vec{\gamma}\cdot\vec{r}} = \vec{\gamma} e^{\vec{\gamma}\cdot\vec{r}} \quad \nabla \cdot \hat{e} e^{\vec{\gamma}\cdot\vec{r}} = \vec{\gamma} \cdot \hat{e} e^{\vec{\gamma}\cdot\vec{r}} \quad \nabla \times \hat{e} e^{\vec{\gamma}\cdot\vec{r}} = \vec{\gamma} \times \hat{e} e^{\vec{\gamma}\cdot\vec{r}}$

Consider a uniform plane wave:

$$\vec{E} = \vec{E}_0 e^{-j\vec{\beta}\cdot\vec{r}} \qquad \vec{H} = \vec{H}_0 e^{-j\vec{\beta}\cdot\vec{r}}$$

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \qquad \nabla \times \vec{H} = j\omega\varepsilon\vec{E}$$

$$-j\vec{\beta} \times \vec{E}_0 e^{-j\vec{\beta}\cdot\vec{r}} = -j\omega\mu\vec{H} \qquad -j\vec{\beta} \times \vec{H}_0 e^{-j\vec{\beta}\cdot\vec{r}} = j\omega\varepsilon\vec{E}$$

$$\vec{\beta} \times \vec{E} = \omega\mu\vec{H} \qquad \vec{\beta} \times \vec{H} = -\omega\varepsilon\vec{E}$$

Uniform Plane Waves (5)

- $$\vec{\beta}\cdot\vec{E}=0, \quad \vec{\beta}\cdot\vec{H}=0$$ ⇨

  For a lossless plane waves, $\vec{E}$ and $\vec{H}$ are always perpendicular to $\vec{\beta}$.

- $$\vec{E}\times(\vec{\beta}\times\vec{E})=\omega\mu\vec{E}\times\vec{H}$$

  Since $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

  $$\vec{E}\times(\vec{\beta}\times\vec{E})=(\vec{E}\cdot\vec{E})\vec{\beta}-(\vec{E}\cdot\vec{\beta})\vec{E}=E^2\vec{\beta}$$

  $$\omega\mu\vec{E}\times\vec{H}=E^2\vec{\beta}$$ ⇨ $\vec{E},\vec{H},\vec{\beta}$ form a triad

- $$\vec{\beta}\times(\vec{\beta}\times\vec{E})=\omega\mu\vec{\beta}\times\vec{H}=-\omega^2\mu\varepsilon\vec{E}=-k^2\vec{E}$$

  Since $\vec{\beta}\times(\vec{\beta}\times\vec{E})=-\vec{\beta}\cdot\vec{\beta}\vec{E}$

  $$\vec{\beta}\cdot\vec{\beta}\vec{E}=k^2\vec{E} \quad \Rightarrow \quad \vec{\beta}\cdot\vec{\beta}=k^2$$ ⇦ Dispersion relation

Uniform Plane Waves (6)

Summary:

$$\beta = \omega\sqrt{\mu\varepsilon}$$

$$Z_w = \frac{E}{H} = \frac{\beta}{\omega\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}} = \eta$$

$$\vec{v}_p = \hat{\beta}\frac{\omega}{\beta} = \hat{\beta}\frac{1}{\sqrt{\mu\varepsilon}}$$

$$\overline{w}_e = \frac{1}{4}\varepsilon\left|\vec{E}_0\right|$$

$$\overline{w}_m = \frac{1}{4}\mu\left|\vec{H}_0\right| = \frac{1}{4}\varepsilon\left|\vec{E}_0\right| = \overline{w}_e$$

$$\vec{S} = \frac{1}{2}\vec{E}\times\vec{H}^* = \frac{\left|E_0\right|^2}{2\omega\mu}\vec{\beta} = \frac{\left|E_0\right|^2}{2\eta}\hat{\beta}$$

Uniform Plane Waves (7)

For a medium with a negative permittivity and negative permeability (Double negative):

$$\vec{\beta}\times\vec{E}=\omega\mu\vec{H}=-\omega|\mu|\vec{H} \qquad \vec{\beta}\times\vec{H}=-\omega\varepsilon\vec{E}=\omega|\varepsilon|\vec{E}$$

$$\vec{E}\times(\vec{\beta}\times\vec{E})=-\omega|\mu|\vec{E}\times\vec{H}$$

Since $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

$$\vec{E}\times(\vec{\beta}\times\vec{E})=(\vec{E}\cdot\vec{E})\vec{\beta}-(\vec{E}\cdot\vec{\beta})\vec{E}=E^2\vec{\beta}$$

$$\omega|\mu|\vec{E}\times\vec{H}=-E^2\vec{\beta}$$

⇒ $\vec{E},\vec{H},\vec{\beta}$ form a left-hand triad

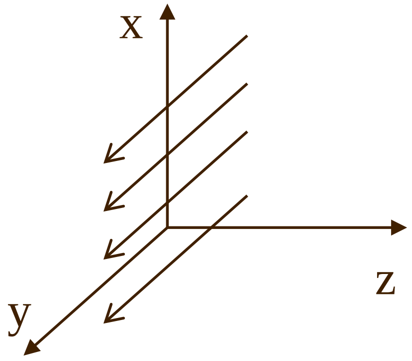
Left-handed material (LHM): Backward propagation

Plane Waves Generated by Current Sheet (1)

Assume a time-harmonic current:

$$\vec{J} = \hat{y} J_0 e^{-jhx} \delta(z) \quad \mathrm{A}/\mathrm{m}^2$$

$$\vec{J}_s = \hat{y} J_0 e^{-jhx} \quad \mathrm{A}/\mathrm{m}$$



Find fields generated by this current sheet.

For $z > 0$:

$$\vec{H}_1 = \vec{H}_{01} e^{\pm j\beta_x x \pm j\beta_y y - j\beta_z z} \qquad \beta_x^2 + \beta_y^2 + \beta_z^2 = \omega^2 \mu_1 \varepsilon_1$$

For $z < 0$:

$$\vec{H}_2 = \vec{H}_{02} e^{\pm j\beta'_x x \pm j\beta'_y y + j\beta'_z z} \qquad \beta'^2_x + \beta'^2_y + \beta'^2_z = \omega^2 \mu_2 \varepsilon_2$$

Plane Waves Generated by Current Sheet (2)

From boundary condition : $\hat{n}\times(\vec{H}_1-\vec{H}_2)=\vec{J}_s$

$$\hat{z}\times\vec{H}_{01}e^{\pm j\beta_x x\pm j\beta_y y}-\hat{z}\times\vec{H}_{02}e^{\pm j\beta'_x x\pm j\beta'_y y}=\hat{y}J_0e^{-jhx}$$

From phase matching : $\beta_y=\beta'_y=0,\ \pm\beta_x=\pm\beta'_x=-h$

$$\vec{H}_1=\vec{H}_{01}e^{-jhx-j\beta_z z} \qquad \beta_z=\sqrt{\omega^2\mu_1\varepsilon_1-h^2}$$

$$\vec{H}_2=\vec{H}_{02}e^{-jhx+j\beta'_z z} \qquad \beta'_z=\sqrt{\omega^2\mu_2\varepsilon_2-h^2}$$

$$\hat{z}\times\vec{H}_{01}-\hat{z}\times\vec{H}_{02}=\hat{y}\vec{J}_0 \Rightarrow \begin{cases} H_{01x}-H_{02x}=J_0 \\ H_{01y}-H_{02y}=0 \end{cases}$$

Next, find the electric field.

Plane Waves Generated by Current Sheet (3)

From $\nabla \times \vec{H} = j\omega\varepsilon\vec{E} \quad \Rightarrow \quad -j\vec{\beta} \times \vec{H} = j\omega\varepsilon\vec{E}$

$$\vec{E}_1 = -\frac{1}{\omega\varepsilon_1}\left(\hat{x}h + \hat{z}\beta_z\right) \times \vec{H}_{01} e^{-jhx - j\beta_z z}$$

$$\vec{E}_2 = -\frac{1}{\omega\varepsilon_2}\left(\hat{x}h - \hat{z}\beta_z'\right) \times \vec{H}_{02} e^{-jhx + j\beta_z' z}$$

Assume that $\mu_1 = \mu_2,\ \varepsilon_1 = \varepsilon_2.$

From boundary condition : $\hat{n} \times (\vec{E}_1 - \vec{E}_2) = 0$

$$hH_{01z} - \beta_z H_{01x} = hH_{02z} + \beta_z H_{02x}$$

$$\beta_z H_{01y} = -\beta_z H_{02y} \quad \Rightarrow \quad H_{01y} = H_{02y} = 0$$

Plane Waves Generated by Current Sheet (4)

$$\nabla \cdot \vec{H} = 0 \quad \Rightarrow \quad \vec{\beta} \cdot \vec{H} = 0 \quad \Rightarrow \quad \begin{cases} hH_{01x} + \beta_z H_{01z} = 0 \\ hH_{02x} - \beta_z H_{02z} = 0 \end{cases}$$

$$\Rightarrow \quad H_{01x} + H_{02x} = 0 \quad \Rightarrow \quad H_{01x} = -H_{02x} = \frac{J_0}{2}$$

Therefore,

$$\vec{H}_1 = \left(\hat{x} - \hat{z}\frac{h}{\beta_z} \right) \frac{J_0}{2} e^{-jhx - j\beta_z z}$$

$$\vec{H}_2 = \left(-\hat{x} - \hat{z}\frac{h}{\beta_z} \right) \frac{J_0}{2} e^{-jhx + j\beta_z z}$$

Special case:

$$h = 0, \quad \vec{H}_1 = \hat{x}\frac{J_0}{2} e^{-j\beta_z z}, \quad \vec{H}_2 = -\hat{x}\frac{J_0}{2} e^{j\beta_z z}$$