ECE 222b
Applied Electromagnetics
Notes Set 3b

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Reflection and Transmission (1)

1. Normal incidence

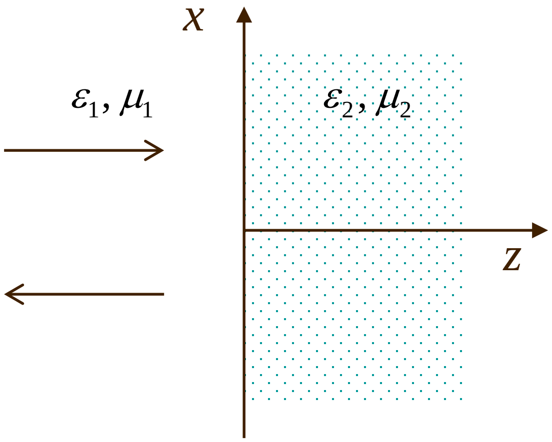
$$\vec{E}^i = \hat{x}E_0 e^{-j\beta_1 z}$$

$$\vec{E}^r = \hat{x}RE_0 e^{j\beta_1 z}$$

$$\vec{E}^t = \hat{x}TE_0 e^{-j\beta_2 z}$$

$$\beta_1 = \omega\sqrt{\mu_1\varepsilon_1}$$

$$\beta_2 = \omega\sqrt{\mu_2\varepsilon_2}$$



First, find the magnetic fields:

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\vec{H}^i = \hat{y}\frac{E_0}{\eta_1} e^{-j\beta_1 z}$$

$$\vec{H}^r = -\hat{y}\frac{RE_0}{\eta_1} e^{j\beta_1 z}$$

$$\vec{H}^t = \hat{y}\frac{TE_0}{\eta_2} e^{-j\beta_2 z}$$

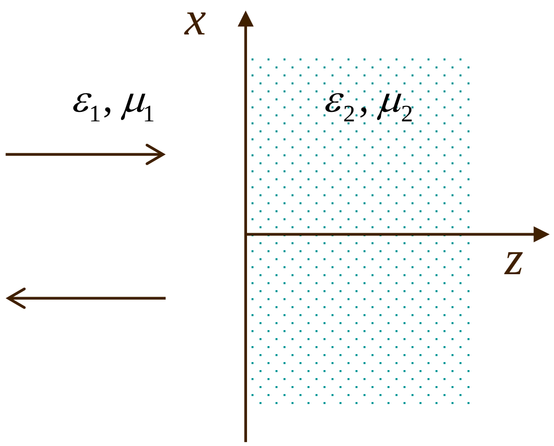
$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

$$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$$

Reflection and Transmission (2)

$$\vec{E}_1 = \vec{E}^i + \vec{E}^r \qquad \vec{E}_2 = \vec{E}^t$$

$$\vec{H}_1 = \vec{H}^i + \vec{H}^r \qquad \vec{H}_2 = \vec{H}^t$$



$$E_0 + RE_0 = TE_0 \quad \Rightarrow \quad 1 + R = T$$

$$\frac{E_0}{\eta_1} - \frac{RE_0}{\eta_1} = \frac{TE_0}{\eta_2} \quad \Rightarrow \quad \frac{1}{\eta_1}(1-R) = \frac{1}{\eta_2}T$$

Solution:

$$R = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}, \quad T = \frac{2\eta_2}{\eta_2 + \eta_1}$$

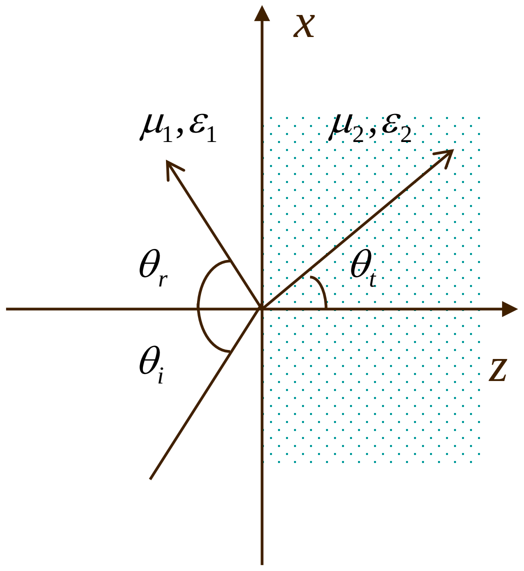
Special case:

1. If $\eta_2 = \eta_1$, then $R = 0, \quad T = 1$
2. If $\eta_2 = 0$, then $R = -1, \quad T = 0$

Reflection and Transmission (3)

2. Oblique incidence

A. Perpendicular polarization (E-pol.)



$$\vec{E}^i = \hat{y}E_0 e^{-j\vec{\beta}^i\cdot\vec{r}} = \hat{y}E_0 e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{E}^r = \hat{y}R_\perp E_0 e^{-j\vec{\beta}^i\cdot\vec{r}} = \hat{y}R_\perp E_0 e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{E}^t = \hat{y}T_\perp E_0 e^{-j\vec{\beta}^t\cdot\vec{r}} = \hat{y}T_\perp E_0 e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$\nabla\times\vec{E} = -j\omega\mu\vec{H}$$

$$\vec{H}^i = (-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)\frac{E_0}{\eta_1}e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\Rightarrow \quad \vec{H}^r = (\hat{x}\cos\theta_r + \hat{z}\sin\theta_r)\frac{R_\perp E_0}{\eta_1}e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}^t = (-\hat{x}\cos\theta_t + \hat{z}\sin\theta_t)\frac{T_\perp E_0}{\eta_2}e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

Reflection and Transmission (4)

$$\hat{n}\times(\vec{E}_1-\vec{E}_2)=0, \quad \text{or} \quad E_{1\tan}=E_{2\tan} \quad \Rightarrow$$

$$E_0 e^{-j\beta_1\sin\theta_i x}+R_\perp E_0 e^{-j\beta_1\sin\theta_r x}=T_\perp E_0 e^{-j\beta_2\sin\theta_t x}$$

$$\hat{n}\times(\vec{H}_1-\vec{H}_2)=0, \quad \text{or} \quad H_{1\tan}=H_{2\tan} \quad \Rightarrow$$

$$-\cos\theta_i\frac{E_0}{\eta_1}e^{-j\beta_1\sin\theta_i x}+\cos\theta_r\frac{R_\perp E_0}{\eta_1}e^{-j\beta_1\sin\theta_r x}=-\cos\theta_t\frac{T_\perp E_0}{\eta_2}e^{-j\beta_2\sin\theta_t x}$$

Phase matching: $\beta_1\sin\theta_i=\beta_1\sin\theta_r=\beta_2\sin\theta_t$

$$\sin\theta_i=\sin\theta_r \quad \Rightarrow \quad \theta_i=\theta_r$$ ⇐ Snell's law of reflection

$$\beta_1\sin\theta_i=\beta_2\sin\theta_t \quad \Rightarrow \quad \frac{\sin\theta_i}{\sin\theta_t}=\sqrt{\frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}}$$ ⇐ Snell's law of refraction

Reflection and Transmission (5)

$$1 + R_\perp = T_\perp \qquad -\cos\theta_i \frac{1}{\eta_1} + \cos\theta_r \frac{R_\perp}{\eta_1} = -\cos\theta_t \frac{T_\perp}{\eta_2}$$

Solution:

$$R_\perp = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \qquad T_\perp = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

$$R_\perp = \frac{\frac{\eta_2}{\cos\theta_t} - \frac{\eta_1}{\cos\theta_i}}{\frac{\eta_2}{\cos\theta_t} + \frac{\eta_1}{\cos\theta_i}} = \frac{Z_{z2} - Z_{z1}}{Z_{z2} + Z_{z1}}$$

$$Z_z = -\frac{E_y}{H_x} = \frac{\eta}{\cos\theta}$$

$$T_\perp = \frac{2\frac{\eta_2}{\cos\theta_t}}{\frac{\eta_2}{\cos\theta_t} + \frac{\eta_1}{\cos\theta_i}} = \frac{2Z_{z2}}{Z_{z2} + Z_{z1}}$$

These formulas are very similar to those in transmission lines!

Reflection and Transmission (6)

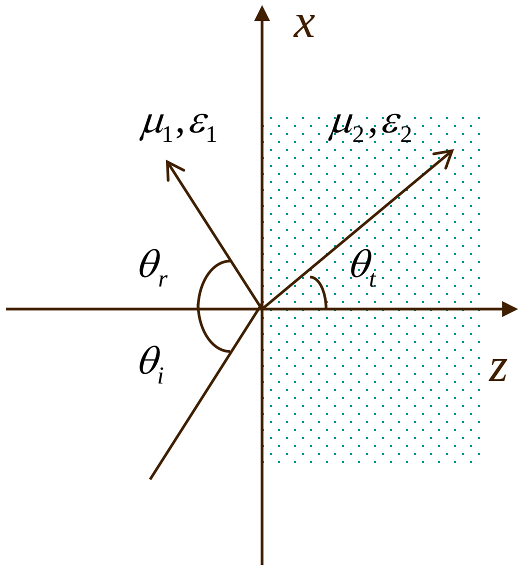
2. Oblique incidence

B. Parallel polarization (H-pol.)

$$\vec{H}^{i} = \hat{y} H_0 e^{-j\beta_1 (x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}^{r} = -\hat{y} R_{\parallel} H_0 e^{-j\beta_1 (x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}^{t} = \hat{y} T_{\parallel} H_0 e^{-j\beta_2 (x\sin\theta_t + z\cos\theta_t)}$$



Result:

$$\theta_r = \theta_i \qquad \frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}}$$

$$R_{\parallel} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

$$T_{\parallel} = \frac{2\eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

Reflection and Transmission (7)

Let $Z_z = \dfrac{E_x}{H_y} = \eta\cos\theta$

$$R_{\parallel} = \frac{Z_{z2} - Z_{z1}}{Z_{z2} + Z_{z1}} \qquad T_{\parallel} = \frac{2Z_{z1}}{Z_{z2} + Z_{z1}}$$

Brewster angle (No reflection)

A. For E-polarization

$$R_{\perp} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

Brewster angle ↓

If $\eta_2 \cos\theta_{iB} = \eta_1 \cos\theta_t$ ⇒ $R_{\perp} = 0$ (Total transmission)

⇓

$$\sqrt{\frac{\mu_2}{\varepsilon_2}}\cos\theta_{iB} = \sqrt{\frac{\mu_1}{\varepsilon_1}}\sqrt{1-\sin^2\theta_t} \qquad \frac{\sin\theta_{iB}}{\sin\theta_t} = \sqrt{\frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}}$$

(Snell's law)

Reflection and Transmission (8)

Solution:

$$\sin\theta_{iB} = \pm\sqrt{\frac{\varepsilon_2/\varepsilon_1 - \mu_2/\mu_1}{\mu_1/\mu_2 - \mu_2/\mu_1}}$$

For $\mu_1 = \mu_2$, no total transmission unless $\varepsilon_1 = \varepsilon_2$!

B. For H-polarization

$$R_{\parallel} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

If $\eta_2 \cos\theta_t = \eta_1 \cos\theta_{iB}$ ⇨ $R_{\parallel} = 0$ (Total transmission)

↑ Brewster angle

$$\sin\theta_{iB} = \pm\sqrt{\frac{\varepsilon_2/\varepsilon_1 - \mu_2/\mu_1}{\varepsilon_2/\varepsilon_1 - \varepsilon_1/\varepsilon_2}}$$

$$\boxed{\sin\theta_{iB} = \pm\sqrt{\frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}}} \quad (\text{if } \mu_1 = \mu_2)$$

Reflection and Transmission (9)

Critical angle (Total reflection):

Snell' s law of refraction

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}} \qquad \sin\theta_t = \sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}}\sin\theta_i$$

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1-\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_i}$$

$$\sin\theta_i = \sqrt{\frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}}, \quad \sin\theta_t = 1 \Rightarrow \theta_t = \frac{\pi}{2}$$

$$\boxed{\theta_c = \sin^{-1}\sqrt{\frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}}} \qquad \theta_c = \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \quad (\text{if } \mu_2 = \mu_1)$$

Critical angle (It exists only if $\sqrt{\mu_2\varepsilon_2} < \sqrt{\mu_1\varepsilon_1}$)

Reflection and Transmission (10)

When $\theta_i = \theta_c$: $\quad R_\perp = 1, \quad T_\perp = 2$

$$\vec{E}^t = \hat{y} T_\perp E_0 e^{-j\beta_2 (x \sin\theta_t + z\cos\theta_t)} = \hat{y} 2 E_0 e^{-j\beta_2 x}$$

$$\vec{H}^t = (-\hat{x}\cos\theta_t + \hat{z}\sin\theta_t)\frac{T_\perp E_0}{\eta_2} e^{-j\beta_2 (x \sin\theta_t + z\cos\theta_t)} = \hat{z}\frac{2E_0}{\eta_2} e^{-j\beta_2 x}$$

$$\vec{S}^t_{av}\Big|_{\theta_i=\theta_c} = \frac{1}{2}\mathrm{Re}(\vec{E}^t \times \vec{H}^{t^*}) = \hat{x}\frac{2|E_0|^2}{\eta_2}$$

When does the power in medium 2 come from???

$$\left|\vec{S}^i_{av}\right|_{\theta_i=\theta_c} = \frac{|E_0|^2}{2\eta_1}$$

$$\left|\vec{S}^r_{av}\right|_{\theta_i=\theta_c} = \frac{|E_0|^2}{2\eta_1}$$

All incident power is reflected back!

Reflection and Transmission (11)

What happens when $\theta_i > \theta_c$?

$$\vec{E}^t = \hat{y} T_\perp E_0 e^{-j\beta_2 (x \sin\theta_t + z\cos\theta_t)}$$

When $\theta_i > \theta_c$, $\sin\theta_t > 1$ ⇒ $\sin\theta_t > 1$

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \pm j\sqrt{\sin^2\theta_t - 1}$$

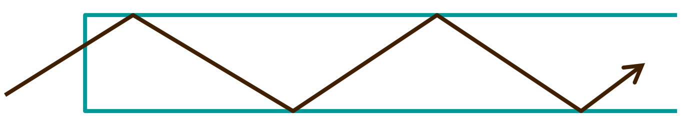
$$\vec{E}^t = \hat{y} T_\perp E_0 e^{-j\beta_2 x \sin\theta_t} \cdot e^{-\beta_2 \sqrt{\sin^2\theta_t - 1}\cdot z}$$

⇐ Non-uniform plane wave

Attenuation constant: $\alpha_e = \beta_2 \sqrt{\sin^2\theta_t - 1}$

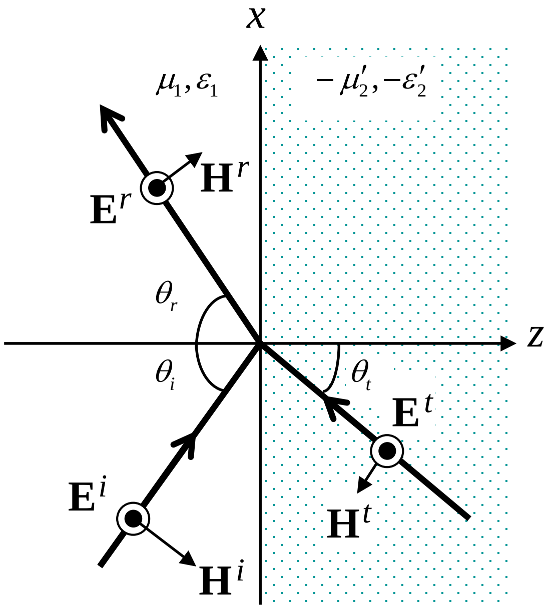
Phase velocity: $v_p = \dfrac{\omega}{\beta_2 \sin\theta_t} < v_{p2}$

Application: Optical fiber



Reflection and Transmission (12)

Transmission into a left-handed medium:



$$\vec{E}^i = \hat{y}E_0 e^{-j\vec{\beta}^i\cdot\vec{r}} = \hat{y}E_0 e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{E}^r = \hat{y}R_\perp E_0 e^{-j\vec{\beta}^r\cdot\vec{r}} = \hat{y}R_\perp E_0 e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{E}^t = \hat{y}T_\perp E_0 e^{-j\vec{\beta}^t\cdot\vec{r}} = \hat{y}T_\perp E_0 e^{-j\beta_2'(x\sin\theta_t - z\cos\theta_t)}$$

$$\vec{H}^i = (-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)\frac{E_0}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}^r = (\hat{x}\cos\theta_r + \hat{z}\sin\theta_r)\frac{R_\perp E_0}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}^t = -(\hat{x}\cos\theta_t + \hat{z}\sin\theta_t)\frac{T_\perp E_0}{\eta_2'} e^{-j\beta_2'(x\sin\theta_t - z\cos\theta_t)}$$

$$\beta_2' = \omega\sqrt{\mu_2'\varepsilon_2'}$$

$$\eta_2' = \sqrt{\mu_2'/\varepsilon_2'}$$

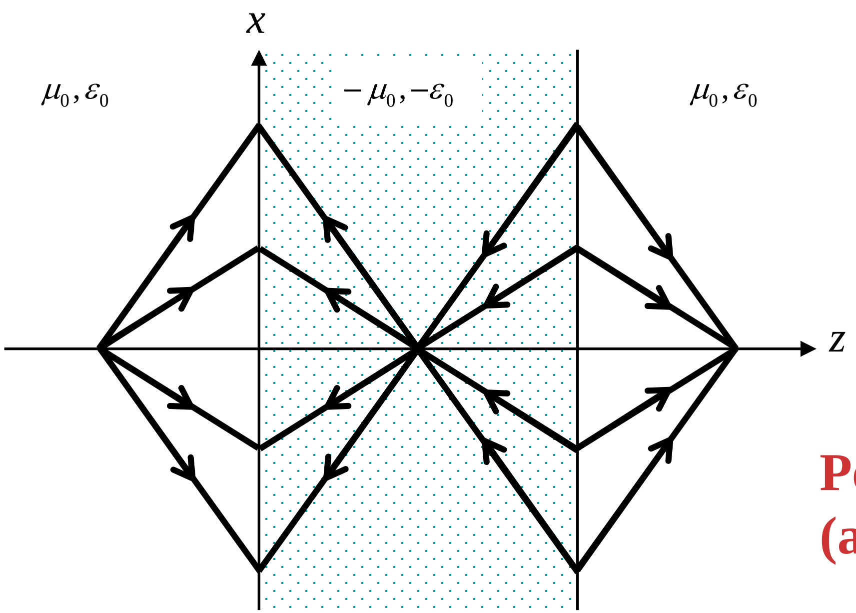
Reflection and Transmission (13)

Phase matching:

$$\sin\theta_t = \sqrt{\frac{\mu_1\varepsilon_1}{\mu_2'\varepsilon_2'}}\sin\theta_i$$

Poynting vector:

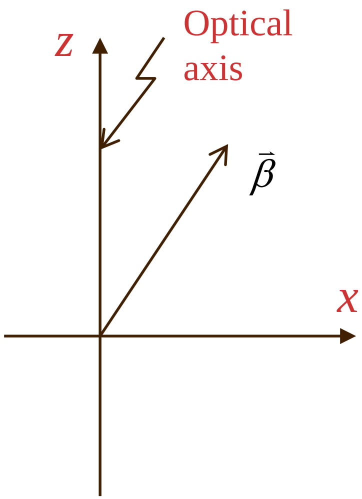
$$\mathbf{S}^t = \frac{1}{2}\mathbf{E}^t \times \mathbf{H}^{t*} = (-\hat{x}\sin\theta_t + \hat{z}\cos\theta_t)\frac{\left|T_\perp E_0\right|^2}{2\eta_2'}$$



Pendry's perfect lens (a plane-wave picture)

Plane Waves in Uniaxial Media (1)

For a uniaxial medium:



$$\overleftrightarrow{\varepsilon} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}$$

$\varepsilon_z > \varepsilon$: positive uniaxial

$\varepsilon_z < \varepsilon$: negative uniaxial

$$\vec{D} = \overleftrightarrow{\varepsilon} \cdot \vec{E}, \qquad \vec{B} = \mu \vec{H}$$

Consider a plane wave: $\vec{E} = \vec{E}_0 e^{-j\vec{\beta}\cdot\vec{r}}$ $\quad \vec{\beta} = \hat{x}\beta_x + \hat{z}\beta_z$

$$\nabla \times \vec{E} = -j\vec{\beta} \times \vec{E} = -j\omega\mu\vec{H} \quad \Rightarrow \quad \vec{\beta} \times \vec{E} = \omega\mu\vec{H}$$

$$\nabla \times \vec{H} = -j\vec{\beta} \times \vec{H} = j\omega\overleftrightarrow{\varepsilon} \cdot \vec{E} \quad \Rightarrow \quad \vec{\beta} \times \vec{H} = -\omega\overleftrightarrow{\varepsilon} \cdot \vec{E}$$

$$\vec{\beta} \times (\vec{\beta} \times \vec{E}) = \omega\mu\vec{\beta} \times \vec{H} = -\omega^2 \mu \overleftrightarrow{\varepsilon} \cdot \vec{E}$$

Plane Waves in Uniaxial Media (2)

$$\vec{\beta}\times\vec{E}=(\hat{x}\beta_x+\hat{z}\beta_z)\times(\hat{x}E_x+\hat{y}E_y+\hat{z}E_z)=\hat{z}\beta_xE_y-\hat{y}\beta_xE_z+\hat{y}\beta_zE_x-\hat{x}\beta_zE_y$$

$$\vec{\beta}\times(\vec{\beta}\times\vec{E})=-\hat{x}\beta_z(\beta_zE_x-\beta_xE_z)-\hat{y}(\beta_x^2E_y+\beta_z^2E_z)+\hat{z}\beta_x(\beta_zE_x-\beta_xE_z)$$

$$=\begin{bmatrix}-\beta_z^2 & 0 & \beta_x\beta_z\\ 0 & -\beta_x^2-\beta_z^2 & 0\\ \beta_x\beta_z & 0 & -\beta_x^2\end{bmatrix}\begin{bmatrix}E_x\\ E_y\\ E_z\end{bmatrix}$$

$$\begin{bmatrix}-\beta_z^2 & 0 & \beta_x\beta_z\\ 0 & -\beta_x^2-\beta_z^2 & 0\\ \beta_x\beta_z & 0 & -\beta_x^2\end{bmatrix}\begin{bmatrix}E_x\\ E_y\\ E_z\end{bmatrix}=-\omega^2\mu\begin{bmatrix}\varepsilon & 0 & 0\\ 0 & \varepsilon & 0\\ 0 & 0 & \varepsilon_z\end{bmatrix}\begin{bmatrix}E_x\\ E_y\\ E_z\end{bmatrix}$$

$$\begin{bmatrix}\beta_z^2-\omega^2\mu\varepsilon & 0 & -\beta_x\beta_z\\ 0 & \beta_x^2+\beta_z^2-\omega^2\mu\varepsilon & 0\\ -\beta_x\beta_z & 0 & \beta_x^2-\omega^2\mu\varepsilon_z\end{bmatrix}\begin{bmatrix}E_x\\ E_y\\ E_z\end{bmatrix}=0$$

Plane Waves in Uniaxial Media (3)

For non-trivial solutions:

$$(\beta_x^2+\beta_z^2-\omega^2\mu\varepsilon)[\,(\beta_x^2-\omega^2\mu\varepsilon_z)\,(\beta_z^2-\omega^2\mu\varepsilon)-\beta_x^2\beta_z^2\,]=0$$

Two possible solutions:

1. $\beta_x^2+\beta_z^2-\omega^2\mu\varepsilon=0 \qquad\qquad E_x=E_z=0$

$$\vec{E}=\hat{y}E_0e^{-j\vec{\beta}\cdot\vec{r}}$$

$$\vec{H}=\frac{1}{\omega\mu}\vec{\beta}\times\vec{E}=\frac{1}{\omega\mu}(-\hat{x}\beta_z+\hat{z}\beta_x)E_0e^{-j\vec{\beta}\cdot\vec{r}}$$

$$\vec{B}=\mu\vec{H}=\frac{1}{\omega}(-\hat{x}\beta_z+\hat{z}\beta_x)E_0e^{-j\vec{\beta}\cdot\vec{r}}$$

$$\vec{D}=\overline{\overline{\varepsilon}}\cdot\vec{E}=\hat{y}\varepsilon E_0e^{-j\vec{\beta}\cdot\vec{r}}$$

This wave behaves the same as those in an isotropic medium with a permittivity ε.

$\Rightarrow$ Ordinary wave

Plane Waves in Uniaxial Media (4)

2. $(\beta_z^2-\omega^2\mu\varepsilon)(\beta_x^2-\omega^2\mu\varepsilon_z)-\beta_x^2\beta_z^2=0$

$$\beta_z^2\omega^2\mu\varepsilon+\beta_x^2\omega^2\mu\varepsilon=\omega^4\mu^2\varepsilon\varepsilon_z$$

$$\boxed{\frac{\beta_x^2}{\omega^2\mu\varepsilon_z}+\frac{\beta_z^2}{\omega^2\mu\varepsilon}=1}$$

Dispersion relation

$$E_y=0 \qquad (\beta_z^2-\omega^2\mu\varepsilon)E_x-\beta_x\beta_z E_z=0$$

$$\frac{\varepsilon}{\varepsilon_z}\beta_x^2E_x+\beta_x\beta_zE_z=0 \;\Rightarrow\; \varepsilon\beta_xE_x+\varepsilon_z\beta_zE_z=0 \;\Rightarrow\; \vec{\beta}\cdot\vec{D}=0$$

$$\vec{E}=(\hat{x}E_x+\hat{z}E_z)=\left(\hat{x}-\hat{z}\frac{\beta_x\varepsilon}{\beta_z\varepsilon_z}\right)E_{x0}e^{-j\vec{\beta}\cdot\vec{r}}$$

$$\vec{H}=\frac{1}{\omega\mu}\vec{\beta}\times\vec{E}=\hat{y}\frac{\omega\varepsilon}{\beta_z}E_{x0}e^{-j\vec{\beta}\cdot\vec{r}} \qquad \vec{D}=\overline{\overline{\varepsilon}}\cdot\vec{E}=\left(\hat{x}-\hat{z}\frac{\beta_x}{\beta_z}\right)\varepsilon E_{x0}e^{-j\vec{\beta}\cdot\vec{r}}$$

$$\vec{B}=\mu\vec{H}=\hat{y}\frac{\omega\mu\varepsilon}{\beta_z}E_{x0}e^{-j\vec{\beta}\cdot\vec{r}}$$

Plane Waves in Uniaxial Media (5)

Observation: $\vec{\beta}\cdot\vec{D}=0,\ \vec{\beta}\cdot\vec{B}=0,\ \vec{\beta}\cdot\vec{H}=0,\ \vec{\beta}\cdot\vec{E}\neq 0$

Thus, $\vec{\beta}$ is not perpendicular to $\vec{E}$. As a result, the Poynting vector $\vec{E}\times\vec{H}$ is not in the direction of $\vec{\beta}$. $\Rightarrow$ Extraordinary wave

Special Cases:

1. If the wave propagates perpendicular to the optical axis:

$$\vec{\beta}=\hat{x}\beta_x, \qquad \beta_y=\beta_z=0$$

For ordinary waves, $\beta_x=\omega\sqrt{\mu\varepsilon}=k_o$

$$\vec{E}=\hat{y}E_0e^{-j\vec{\beta}\cdot\vec{r}}=\hat{y}E_0e^{-j\beta_x x}=\hat{y}E_0e^{-jk_o x}$$

For extraordinary waves, $\beta_x=\omega\sqrt{\mu\varepsilon_z}=k_e$

$$\vec{E}=\hat{z}E_0e^{-j\vec{\beta}\cdot\vec{r}}=\hat{z}E_0e^{-j\beta_x x}=\hat{z}E_0e^{-jk_e x}$$

Plane Waves in Uniaxial Media (6)

Phase velocity:

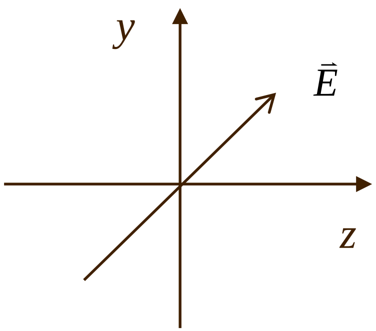
$$v_{po} = \frac{\omega}{k_o} = \frac{1}{\sqrt{\mu\varepsilon}}$$

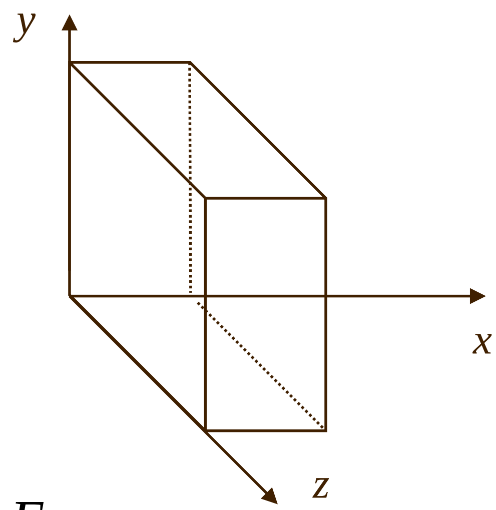
$$v_{pe} = \frac{\omega}{k_e} = \frac{1}{\sqrt{\mu\varepsilon_e}}$$

} Birefringence

At the front: (Linearly polarized)

$$\vec{E} = \left(\hat{y}\frac{1}{\sqrt{2}} + \hat{z}\frac{1}{\sqrt{2}}\right)E_0 e^{-j\beta_x x}$$





After passing through the slab:

$$\vec{E} = \hat{y}\frac{E_0}{\sqrt{2}}e^{-jk_o d} + \hat{z}\frac{E_0}{\sqrt{2}}e^{-jk_e d} = \left(\hat{y} + \hat{z}e^{-j(k_e - k_o)d}\right)\frac{E_0}{\sqrt{2}}e^{-jk_o d}$$

Plane Waves in Uniaxial Media (7)

If $(k_e - k_o)d = \pm\dfrac{\pi}{2},\ \pm\dfrac{3\pi}{2},\cdots$

$$\vec{E} = (\hat{y} \pm j\hat{z})\frac{E_0}{\sqrt{2}}e^{-jk_o d}$$

(circularly polarized)

If $\left|(k_e - k_o)d\right| = \dfrac{\pi}{2},\quad d = \dfrac{\pi}{2\left|k_e - k_o\right|},\quad k_e = \dfrac{2\pi}{\lambda_e},\ k_o = \dfrac{2\pi}{\lambda_o},$

$$\left|k_e - k_o\right| \equiv \frac{2\pi}{\lambda_d},\quad \lambda_d = \frac{\lambda_e \lambda_o}{\left|\lambda_e - \lambda_o\right|},\quad d = \frac{\lambda_d}{4}$$

Quarter-wave plate

Plane Waves in Uniaxial Media (8)

2. Polaroid

$$\overleftrightarrow{\varepsilon} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon_z - j\frac{\sigma_z}{\omega} \end{bmatrix}$$

For ordinary waves ($\vec{E}$ perpendicular to the optical axis): $k_o = \omega\sqrt{\mu\varepsilon}$

For extraordinary waves ($\vec{E}$ parallel to the optical axis):

$$k_e = \omega\sqrt{\mu\left(\varepsilon_z - j\frac{\sigma_z}{\omega}\right)} \sim \sqrt{\frac{\omega\mu\sigma}{2}} - j\sqrt{\frac{\omega\mu\sigma}{2}}$$

Extraordinary waves will be attenuated.

Similarly, $\overline{\overline{\varepsilon}} = \begin{bmatrix} \varepsilon - j\frac{\sigma_z}{\omega} & 0 & 0 \\ 0 & \varepsilon - j\frac{\sigma_z}{\omega} & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \Rightarrow$ E_z passes through, E_y is attenuated

Plane Waves in Gyrotropic Media (1)

For a gyrotropic medium:

$$\overleftrightarrow{\varepsilon} = \begin{bmatrix} \varepsilon & -j\varepsilon_g & 0 \\ j\varepsilon_g & \varepsilon & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}$$

Consider a plane wave: $\vec{E} = \vec{E}_0 e^{-j\vec{\beta}\cdot\vec{r}}$ $\qquad \vec{\beta} = \hat{x}\beta_x + \hat{z}\beta_z$

We find $\vec{\beta}\times(\vec{\beta}\times\vec{E}) = -\omega^2\mu\overleftrightarrow{\varepsilon}\cdot\vec{E}$ $\Rightarrow$

$$\begin{bmatrix} \beta_z^2 - \omega^2\mu\varepsilon & j\omega^2\mu\varepsilon_g & -\beta_x\beta_z \\ -j\omega^2\mu\varepsilon_g & \beta_x^2 + \beta_z^2 - \omega^2\mu\varepsilon & 0 \\ -\beta_x\beta_z & 0 & \beta_x^2 - \omega^2\mu\varepsilon_z \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0$$

Plane Waves in Gyrotropic Media (2)

For non-trivial solutions:

$$(\beta_x^2 + \beta_z^2 - \omega^2 \mu\varepsilon)(-\omega^2 \mu\varepsilon\beta_x^2 - \omega^2 \mu\varepsilon_z\beta_z^2 + \omega^4 \mu^2 \varepsilon\varepsilon_z)$$
$$- \omega^4 \mu^2 \varepsilon_g^2 (\beta_x^2 - \omega^2 \mu\varepsilon_z) = 0$$

Rewrite: $\vec{\beta} = \hat{x}\beta_x + \hat{z}\beta_z = \hat{x}\beta \sin\theta + \hat{z}\beta\cos\theta$

Define: $K = \omega\sqrt{\mu\varepsilon}, \quad K_z = \omega\sqrt{\mu\varepsilon_z}, \quad K_g = \omega\sqrt{\mu\varepsilon_g}$

$$A\beta^4 - B\beta^2 + C = 0$$

$$A = K^2 \sin^2\theta + K_z^2 \cos^2\theta$$

$$B = (K^2 - K_g^2)\sin^2\theta + K^2 K_z^2 (1 + \cos^2\theta)$$

$$C = (K^4 - K_g^4) K_z^2$$

Plane Waves in Gyrotropic Media (3)

Two solutions:

$$\beta^2 = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$\begin{bmatrix} \beta^2\cos^2\theta - K^2 & jK_g^2 & -\beta^2\sin\theta\cos\theta \\ -jK_g^2 & \beta^2 - K^2 & 0 \\ -\beta^2\sin\theta\cos\theta & 0 & \beta^2\sin^2\theta - K_z^2 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0$$

$$\Rightarrow \quad \frac{E_x}{E_y} = \frac{\beta^2 - K^2}{jK_g^2} \qquad \frac{E_x}{E_z} = \frac{\beta^2\sin^2\theta - K_z^2}{\beta^2\sin\theta\cos\theta}$$

Special cases:

(1) $\theta = 0$ $\quad \vec{\beta} = \hat{z}\beta$

$$\begin{bmatrix} \beta^2 - K^2 & jK_g^2 & 0 \\ -jK_g^2 & \beta^2 - K^2 & 0 \\ 0 & 0 & -K_z^2 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0$$

Plane Waves in Gyrotropic Media (4)

$$E_z = 0 \qquad (\beta^2 - K^2)^2 - K_g^4 = 0$$

$$(\beta^2 - K^2 + K_g^2)(\beta^2 - K^2 - K_g^2) = 0$$

$$\beta_\pm^2 = K^2 \pm K_g^2 = \omega^2 \mu(\varepsilon \pm \varepsilon_g)$$

$$\frac{E_x}{E_y} = \frac{\beta^2 - K^2}{jK_g^2} = \frac{\pm K_g^2}{jK_g^2} = \mp j$$

(a) $\dfrac{E_x}{E_y} = -j \qquad \vec{E} = \hat{x}E_x + \hat{y}E_y = (\hat{x} + j\hat{y})E_x$

$\Rightarrow$ left-hand circularly polarized wave

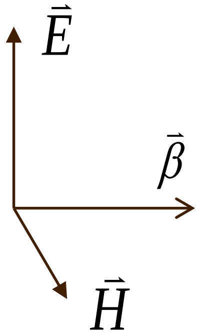
$$\beta_+ = \omega\sqrt{\mu(\varepsilon + \varepsilon_g)} \qquad v_{p+} = \frac{\omega}{\beta_+} = \frac{1}{\sqrt{\mu(\varepsilon + \varepsilon_g)}}$$

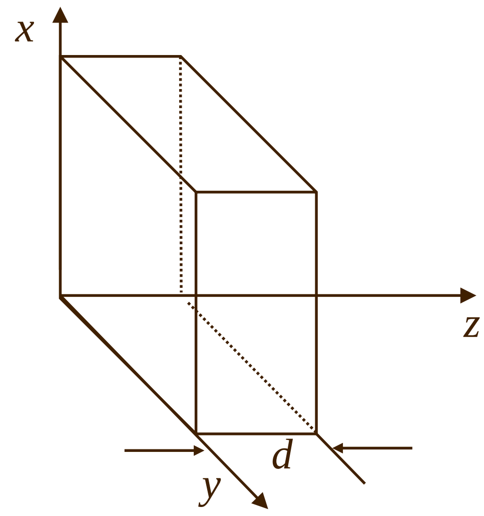
Plane Waves in Gyrotropic Media (5)

(b) $\frac{E_x}{E_y} = j \qquad \vec{E} = \hat{x}E_x + \hat{y}E_y = (\hat{x} - j\hat{y})E_x$

$\Rightarrow$ right - hand circularly polarized wave

$$\beta_- = \omega\sqrt{\mu(\varepsilon - \varepsilon_g)} \qquad v_{p-} = \frac{\omega}{\beta_-} = \frac{1}{\sqrt{\mu(\varepsilon - \varepsilon_g)}}$$





Consider

$\vec{E} = \hat{x}E_0e^{-j\beta z}$ (Linearly polarized)

$$\vec{E} = \underbrace{\frac{1}{2}(\hat{x} - j\hat{y})E_0e^{-j\beta z}}_{RHCP} + \underbrace{\frac{1}{2}(\hat{x} + j\hat{y})E_0e^{-j\beta z}}_{LHCP}$$

Plane Waves in Gyrotropic Media (6)

At $z = d$:

$$\vec{E} = \frac{1}{2}(\hat{x} - j\hat{y})E_0 e^{-j\beta_- d} + \frac{1}{2}(\hat{x} + j\hat{y})E_0 e^{-j\beta_+ d}$$

$$= \frac{1}{2}\hat{x}E_0(e^{-j\beta_- d} + e^{-j\beta_+ d}) - \frac{1}{2}j\hat{y}E_0(e^{-j\beta_- d} - e^{-j\beta_+ d})$$

$$\frac{E_x}{E_y} = \frac{\frac{1}{2}E_0(e^{-j\beta_- d} + e^{-j\beta_+ d})}{\frac{1}{2}jE_0(e^{-j\beta_- d} - e^{-j\beta_+ d})} = j\frac{e^{j(\beta_+ - \beta_-)d} + 1}{e^{j(\beta_+ - \beta_-)d} - 1}$$

$$= j\frac{e^{j(\beta_+ - \beta_-)d/2} + e^{-j(\beta_+ - \beta_-)d/2}}{e^{j(\beta_+ - \beta_-)d/2} - e^{-j(\beta_+ - \beta_-)d/2}} = \cot\left[\frac{(\beta_+ - \beta_-)d}{2}\right]$$

$\Rightarrow$ Still a linearly polarized wave with an angle $\theta_F = \frac{(\beta_+ - \beta_-)d}{2}$

Faraday rotation

Plane Waves in Gyrotropic Media (7)

(2) $\theta = \frac{\pi}{2}, \vec{\beta} = \hat{x}\beta$

$$\begin{bmatrix} -K^2 & jK_g^2 & 0 \\ -jK_g^2 & \beta^2 - K^2 & 0 \\ 0 & 0 & \beta^2 - K_z^2 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0$$

(a) $\beta^2 - K_z^2 = 0, \quad E_z \neq 0, \quad E_x = E_y = 0$

$$\beta = \omega\sqrt{\mu\varepsilon_z}, \quad v_p = \frac{1}{\sqrt{\mu\varepsilon_z}}$$

(b) $K^2(\beta^2 - K^2) + K_g^4 = 0 \quad \Rightarrow \quad \beta^2 = K^2 - \frac{K_g^4}{K^2}$

$$E_z = 0, \quad -K^2 E_x + jK_g^2 E_y = 0$$ ⇦ Elliptically polarized.

Plane Waves in Chiral Media (1)

Constitutive relations:

$$\vec{D} = \varepsilon \vec{E} - j\chi \vec{H}, \qquad \vec{B} = \mu \vec{H} + j\chi \vec{E}$$

χ - chirality parameter

Maxwell's equations:

$$\vec{\beta} \times \vec{E} = \omega \vec{B} \qquad \vec{\beta} \times \vec{H} = -\omega \vec{D}$$

$$\vec{\beta} \cdot \vec{D} = 0 \qquad \vec{\beta} \cdot \vec{B} = 0$$

$$\begin{bmatrix} \vec{\beta} \times \overleftrightarrow{I} - j\omega\chi \overleftrightarrow{I} & -\omega\mu \overleftrightarrow{I} \\ \omega\varepsilon \overleftrightarrow{I} & \vec{\beta} \times \overleftrightarrow{I} - j\omega\chi \overleftrightarrow{I} \end{bmatrix} \cdot \begin{bmatrix} \vec{E} \\ \vec{H} \end{bmatrix} = 0$$

Plane Waves in Chiral Media (2)

Assume $\vec{\beta} = \hat{z}\beta$**:** $D_z = B_z = 0$ $E_z = H_z = 0$

$$\begin{bmatrix} -\beta & j\omega\chi \\ j\omega\chi & \beta \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = -\begin{bmatrix} 0 & \omega\mu \\ \omega\mu & 0 \end{bmatrix} \begin{bmatrix} H_x \\ H_y \end{bmatrix}$$

$$\begin{bmatrix} -\beta & j\omega\chi \\ j\omega\chi & \beta \end{bmatrix} \begin{bmatrix} H_x \\ H_y \end{bmatrix} = \begin{bmatrix} 0 & \omega\varepsilon \\ \omega\varepsilon & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

$$\begin{bmatrix} 2j\omega\chi\beta & \beta^2 + \omega^2\chi^2 - \omega^2\mu\varepsilon \\ \beta^2 + \omega^2\chi^2 - \omega^2\mu\varepsilon & -2j\omega\chi\beta \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = 0$$

Solutions:

$$\beta_{\pm} = \omega\sqrt{\mu\varepsilon} \pm \omega\chi \qquad \frac{E_x}{E_y} = \pm j$$

Plane Waves in Chiral Media (3)

LHCP:

$$\frac{E_x}{E_y} = +j \qquad v_{p+} = \frac{\omega}{\beta_+} = \frac{1}{\sqrt{\mu\varepsilon} + \chi}$$

RHCP:

$$\frac{E_x}{E_y} = -j \qquad v_{p-} = \frac{\omega}{\beta_-} = \frac{1}{\sqrt{\mu\varepsilon} - \chi}$$

Observation: Faraday rotation, birefringence, reciprocal gyrotropic medium.