ECE 222b
Applied Electromagnetics
Notes Set 4b

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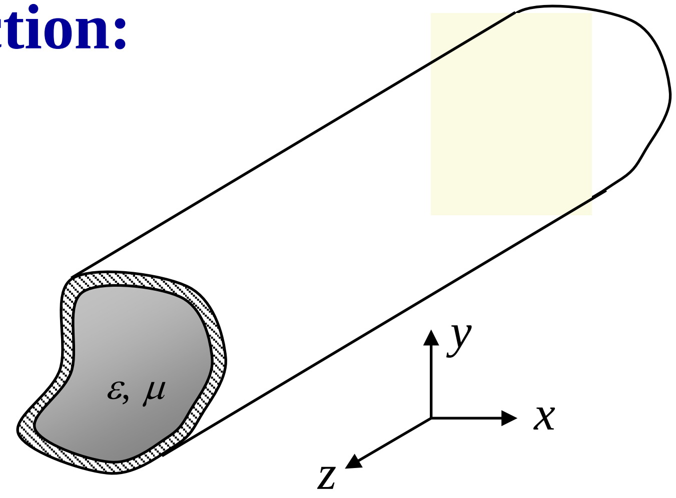
Uniform Waveguide (1)

Wave propagation in the +z direction:

$$\mathbf{E} = \mathbf{E}_t + \hat{z}E_z = \left[\mathbf{e}_t(x,y) + \hat{z}e_z(x,y)\right]e^{-jk_z z}$$

$$\mathbf{H} = \mathbf{H}_t + \hat{z}H_z = \left[\mathbf{h}_t(x,y) + \hat{z}h_z(x,y)\right]e^{-jk_z z}$$



Maxwell' s equations:

$$\left(\nabla_t + \hat{z}\frac{\partial}{\partial z}\right)\times(\mathbf{E}_t + \hat{z}E_z) = -j\omega\mu(\mathbf{H}_t + \hat{z}H_z)$$

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E}$$

$$\hat{z}\times\nabla_t E_z + jk_z\hat{z}\times\mathbf{E}_t = j\omega\mu\mathbf{H}_t$$

$$\nabla_t \times \mathbf{E}_t = -j\omega\mu H_z\hat{z}$$

$$\hat{z}\times\nabla_t H_z + jk_z\hat{z}\times\mathbf{H}_t = -j\omega\varepsilon\mathbf{E}_t$$

$$\nabla_t \times \mathbf{H}_t = j\omega\varepsilon E_z\hat{z}$$

Uniform Waveguide (2)

Transverse field components:

$$\mathbf{E}_t = \frac{1}{k_t^2}(j\omega\mu\hat{z}\times\nabla_t H_z - jk_z\nabla_t E_z)$$

$$\mathbf{H}_t = \frac{1}{k_t^2}(-j\omega\varepsilon\hat{z}\times\nabla_t E_z - jk_z\nabla_t H_z)$$

We only have to deal with the longitudinal components!

$$k_t^2 = k^2 - k_z^2 \qquad k^2 = \omega^2\mu\varepsilon$$

Equations for the longitudinal components:

$$\nabla_t \times \left[\frac{1}{k_t^2}(\omega\varepsilon\hat{z}\times\nabla_t E_z + k_z\nabla_t H_z)\right] = -\omega\varepsilon E_z\hat{z}$$

$$\nabla_t \times \left[\frac{1}{k_t^2}(\omega\mu\hat{z}\times\nabla_t H_z - k_z\nabla_t E_z)\right] = -\omega\mu H_z\hat{z}$$

Uniform Waveguide (3)

Consider two cases:

Case A: Homogeneously filled waveguide:

$k_t^2 = k^2 - k_z^2$ **is independent of *x* and *y***

$$\nabla_t^2 E_z + k_t^2 E_z = 0 \qquad \text{in } \Omega$$

$$\nabla_t^2 H_z + k_t^2 H_z = 0 \qquad \text{in } \Omega$$

What is the implication???

E_z and H_z can exist independently!!!

$E_z \neq 0\ ,\quad H_z = 0$ ⇨ **TM modes**

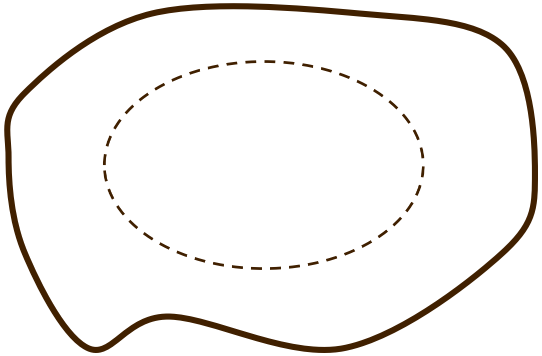
$E_z = 0\ ,\quad H_z \neq 0$ ⇨ **TE modes**

Uniform Waveguide (4)

Why TM or TE? Why not TEM?

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -j\omega\mu \iint_S \mathbf{H}\cdot d\mathbf{S}$$

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = j\omega\varepsilon \iint_S \mathbf{E}\cdot d\mathbf{S} + \mathbf{J}$$



TEM can only exist in waveguides made of two separate conductors. Example: coaxial waveguide.

What's the consequence of have either E_z or H_z?

$\mathbf{S} = \frac{1}{2}\mathbf{E}\times\mathbf{H}^*$ **is not directly in the *z* direction. Zigzag propagation.**

Phase velocity is greater than *c*. Energy velocity is smaller than *c*.

What happens at cutoff?

Uniform Waveguide (5)

Case B: Inhomogeneously filled waveguide:

$k_t^2 = k^2 - k_z^2$ **is a function of *x* and *y***

$$\nabla_t \times \left[\frac{1}{k_t^2} (\omega\varepsilon \hat{z} \times \nabla_t E_z + k_z \nabla_t H_z) \right] = -\omega\varepsilon E_z \hat{z}$$

$$\nabla_t \times \left[\frac{1}{k_t^2} (\omega\mu \hat{z} \times \nabla_t H_z - k_z \nabla_t E_z) \right] = -\omega\mu H_z \hat{z}$$

Consequence: E_z and H_z must co-exist: ⇨ Hybrid modes.

What if the filling is piecewise homogeneous?

E_z and H_z are coupled by the discontinuous interface in order to satisfy the boundary conditions.

Uniform Waveguide (6)

To analyze TM modes:

Solve

$$\nabla_t^2 E_z + k_t^2 E_z = 0 \quad \text{in } \Omega$$
$$E_z = 0 \quad \text{on } \Gamma$$

then

$$\mathbf{E}_t = -\frac{jk_z}{k_t^2}\nabla_t E_z$$
$$\mathbf{H}_t = -\frac{j\omega\varepsilon}{k_t^2}\hat{z}\times\nabla_t E_z$$

To analyze TE modes:

Solve

$$\nabla_t^2 H_z + k_t^2 H_z = 0 \quad \text{in } \Omega$$
$$\frac{\partial H_z}{\partial n} = 0 \quad \text{on } \Gamma$$

then

$$\mathbf{E}_t = \frac{j\omega\mu}{k_t^2}\hat{z}\times\nabla_t H_z$$
$$\mathbf{H}_t = -\frac{jk_z}{k_t^2}\nabla_t H_z$$

Uniform Waveguide (7)

General characteristics:

Propagation constant:

$$k_z = \begin{cases} \sqrt{\omega^2\mu\varepsilon - k_t^2} & \omega\sqrt{\mu\varepsilon} > k_t \\ 0 & \omega\sqrt{\mu\varepsilon} = k_t \\ -j\sqrt{k_t^2 - \omega^2\mu\varepsilon} & \omega\sqrt{\mu\varepsilon} < k_t \end{cases}$$

⟸ **Propagation**

⟸ **Cutoff** $k_c = k_t$

⟸ **Attenuation**

Phase, group, and energy velocities:

$$v_p = \frac{\omega}{k_z} = \frac{c}{\sqrt{1-(k_c/k)^2}} \qquad v_g = \left(\frac{dk_z}{d\omega}\right)^{-1} = c\sqrt{1-\left(\frac{k_c}{k}\right)^2}$$

$$v_e = \frac{\overline{P}_z}{\overline{W}_e + \overline{W}_m} = c\sqrt{1-\left(\frac{k_c}{k}\right)^2}$$

Uniform Waveguide (8)

Guided wavelength:

$$\lambda_g = \frac{2\pi}{k_z} = \frac{\lambda}{\sqrt{1-(k_c/k)^2}}$$

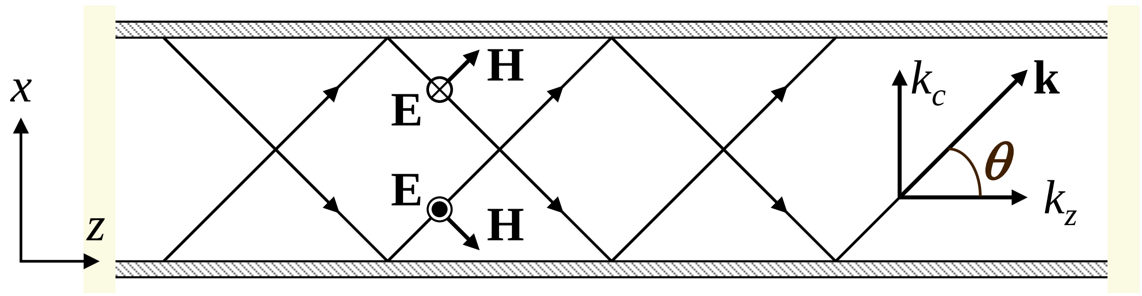
Wave impedance in the *z*-direction:

$$Z_w^{\mathrm{TM}} = \frac{k_z}{\omega\varepsilon} = \eta\sqrt{1-\left(\frac{k_c}{k}\right)^2} = \begin{cases} \text{Resistive} & f > f_c \\ 0 & f = f_c \\ \text{Capacitive} & f < f_c \end{cases}$$

$$Z_w^{\mathrm{TE}} = \frac{\omega\mu}{k_z} = \frac{\eta}{\sqrt{1-(k_c/k)^2}} = \begin{cases} \text{Resistive} & f > f_c \\ \infty & f = f_c \\ \text{Inductive} & f < f_c \end{cases}$$

Uniform Waveguide (9)

What' s the meaning of $\sqrt{1-\left(\frac{k_c}{k}\right)^2}$ **?**



$$\sin\theta = \frac{k_c}{k}$$

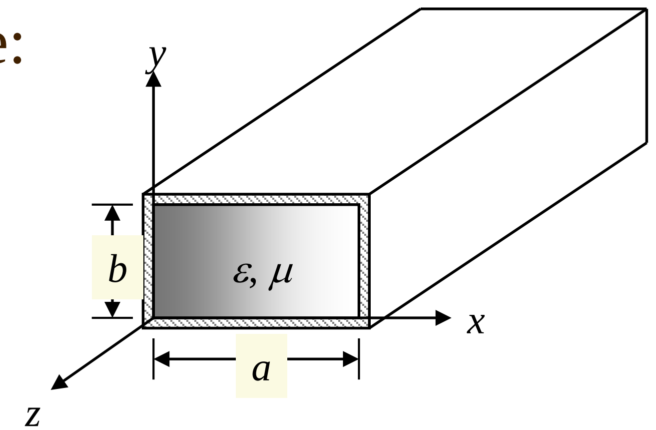
$$\cos\theta = \sqrt{1-\left(\frac{k_c}{k}\right)^2}$$

Mode orthogonality:

1. **All TM modes are orthogonal to each other**
2. **All TE modes are orthogonal to each other**
3. **TM and TE modes are mutually orthogonal**

Rectangular Waveguide (1)

An empty, infinitely long waveguide:



1. TE_z modes

$$\nabla_t^2 H_z + k_t^2 H_z = 0$$

or

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + k_t^2 H_z = 0$$

Let

$$H_z(x, y, z) = f(x)\, g(y)\, e^{-jk_z z}$$

$$g\frac{\partial^2 f}{\partial x^2} + f\frac{\partial^2 g}{\partial y^2} + k_t^2 fg = 0$$

Rectangular Waveguide (2)

$$\frac{1}{f}\frac{\partial^2 f}{\partial x^2}+\frac{1}{g}\frac{\partial^2 g}{\partial y^2}+k_t^2=0$$

$$\frac{1}{f}\frac{\partial^2 f}{\partial x^2}=-k_x^2\,, \qquad \frac{1}{g}\frac{\partial^2 g}{\partial y^2}=-k_y^2 \qquad k_x^2+k_y^2=k_t^2$$

Consider $\frac{1}{f}\frac{\partial^2 f}{\partial x^2}=-k_x^2 \quad \Rightarrow \quad \frac{\partial^2 f}{\partial x^2}+k_x^2 f=0$

Two independent solutions: $e^{jk_x x},\quad e^{-jk_x x}$ or $\sin k_x x\,,\ \cos k_x x$

$$f(x)=C_1\cos k_x x+D_1\sin k_x x$$

Similarly: $g(y)=C_2\cos k_y y+D_2\sin k_y y$

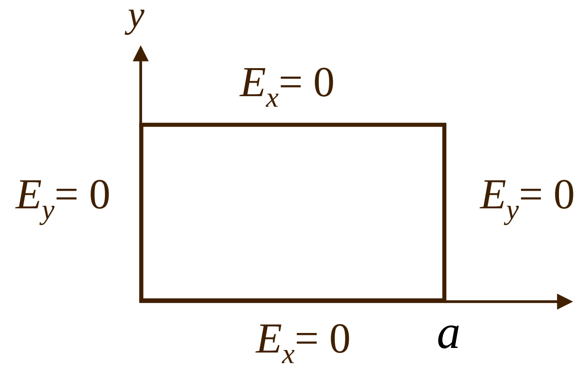
Rectangular Waveguide (3)

General solution:

$$H_z(x, y, z) = \left(C_1 \cos k_x x + D_1 \sin k_x x\right)\left(C_2 \cos k_y y + D_2 \sin k_y y\right) e^{-jk_z z}$$

Boundary condition:

$$\frac{\partial H_z}{\partial n} = 0$$



$$\left.\frac{\partial H_z}{\partial x}\right|_{x=0} = 0 \quad \Rightarrow \quad D_1 = 0$$

$$\left.\frac{\partial H_z}{\partial x}\right|_{x=a} = 0 \quad \Rightarrow \quad \sin k_x a = 0 \quad \Rightarrow \quad k_x a = m\pi$$

$$k_x = \frac{m\pi}{a} \qquad m = 0, 1, 2, \ldots$$

Rectangular Waveguide (4)

$$\left.\frac{\partial H_z}{\partial y}\right|_{y=0} = 0 \quad \Rightarrow \quad D_2 = 0$$

$$\left.\frac{\partial H_z}{\partial y}\right|_{y=b} = 0 \quad \Rightarrow \quad \sin k_y b = 0 \quad \Rightarrow \quad k_y b = n\pi$$

$$k_y = \frac{n\pi}{b} \qquad n = 0, 1, 2, \ldots$$

$$E_x = H_{mn} \frac{j\omega\mu}{k_{tmn}^2} \frac{n\pi}{b} \cos\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) e^{-jk_{zmn} z}$$

$$E_y = -H_{mn} \frac{j\omega\mu}{k_{tmn}^2} \frac{m\pi}{a} \sin\left(\frac{m\pi}{a} x\right) \cos\left(\frac{n\pi}{b} y\right) e^{-jk_{zmn} z}$$

$$E_z = 0$$

Rectangular Waveguide (5)

$$H_x = H_{mn} \frac{jk_{zmn}}{k_{tmn}^2} \frac{m\pi}{a} \sin\left(\frac{m\pi}{a} x\right) \cos\left(\frac{n\pi}{b} y\right) e^{-jk_{zmn} z}$$

$$H_y = H_{mn} \frac{jk_{zmn}}{k_{tmn}^2} \frac{n\pi}{b} \cos\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) e^{-jk_{zmn} z}$$

$$H_z = H_{mn} \cos\left(\frac{m\pi}{a} x\right) \cos\left(\frac{n\pi}{b} y\right) e^{-jk_{zmn} z}$$

Consider k_z :

$$k_{tmn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

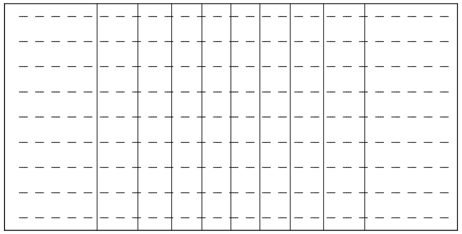
$$k_{zmn}^2 = k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$$

$m, n = 0,1,2,\ldots$

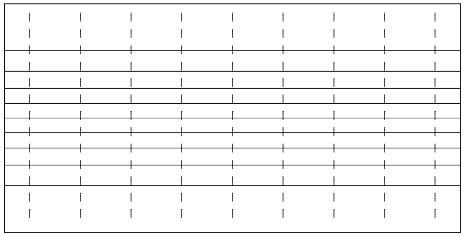
except for $m = n = 0$

Rectangular Waveguide (6)

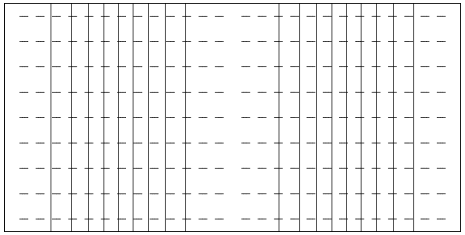
TE modal field distribution:



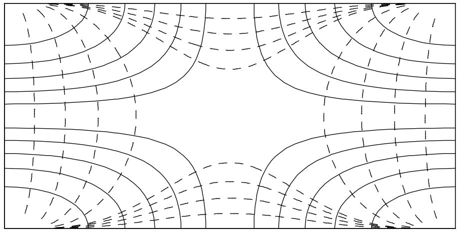
TE_{10}



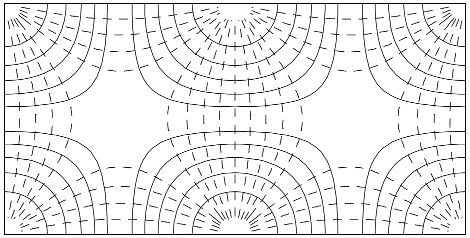
TE_{01}



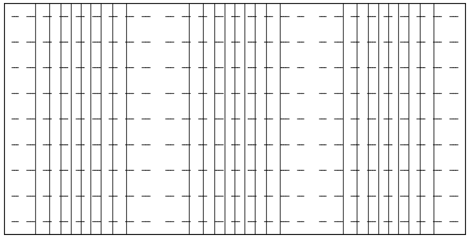
TE_{20}



TE_{11}



TE_{21}



TE_{30}

Rectangular Waveguide (7)

2. TM_z modes

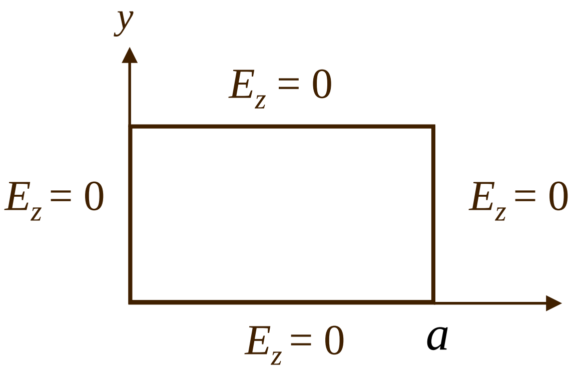
$$\nabla_t^2 E_z + k_t^2 E_z = 0$$

General solution:

$$E_z(x, y, z) = \left(C_1 \cos k_x x + D_1 \sin k_x x\right)\left(C_2 \cos k_y y + D_2 \sin k_y y\right) e^{-jk_z z}$$

Boundary condition:

$$E_z = 0$$



$$E_z\big|_{x=0} = 0 \quad \Rightarrow \quad C_1 = 0$$

Rectangular Waveguide (8)

$$E_z\big|_{x=a}=0 \Rightarrow \sin k_x a = 0 \Rightarrow k_x a = m\pi$$

$$k_x = \frac{m\pi}{a} \qquad m = 1, 2, \ldots$$

$$E_z\big|_{y=0}=0 \Rightarrow C_2 = 0$$

$$E_z\big|_{y=b}=0 \Rightarrow \sin k_y b = 0 \Rightarrow k_y b = n\pi$$

$$k_y = \frac{n\pi}{b} \qquad n = 1, 2, \ldots$$

$$E_z = E_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-jk_{zmn}z}$$

Rectangular Waveguide (9)

$$E_x = -E_{mn} \frac{jk_{zmn}}{k_{tmn}^2} \frac{m\pi}{a} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-jk_{zmn}z}$$

$$E_y = -E_{mn} \frac{jk_{zmn}}{k_{tmn}^2} \frac{n\pi}{b} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-jk_{zmn}z}$$

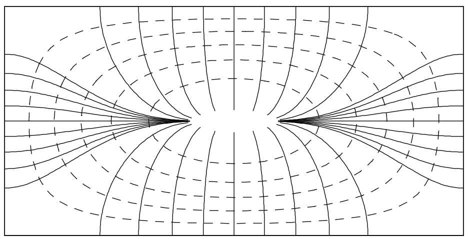
$$H_x = E_{mn} \frac{j\omega\varepsilon}{k_{tmn}^2} \frac{n\pi}{b} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-jk_{zmn}z}$$

$$H_y = -E_{mn} \frac{j\omega\varepsilon}{k_{tmn}^2} \frac{m\pi}{a} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-jk_{zmn}z}$$

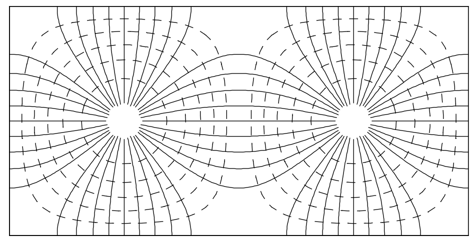
$$H_z = 0$$

Rectangular Waveguide (10)

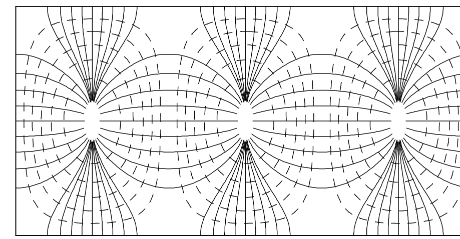
TM modal field distribution:



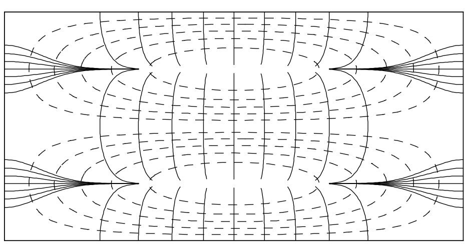
TM_{11}



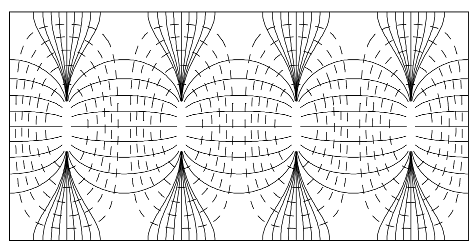
TM_{21}



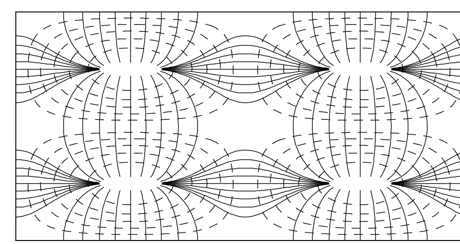
TM_{31}



TM_{12}



TM_{41}



TM_{22}

Rectangular Waveguide (11)

$$k_{zmn} = \begin{cases} \sqrt{k^2 - (\frac{m\pi}{a})^2 - (\frac{n\pi}{b})^2} & k^2 > (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 \\ 0 & k^2 = (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 \\ -j\sqrt{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 - k^2} & k^2 < (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 \end{cases}$$

Define $k_{cmn} = \sqrt{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2}$ cutoff wavenumber

$$f_{cmn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2}$$ cutoff frequency

$$\lambda_{cmn} = \frac{2\pi}{k_{cmn}} = \frac{2\pi}{\sqrt{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2}}$$ cutoff wavelength

Rectangular Waveguide (12)

Guide wavelength:

$$k_{zmn} = \frac{2\pi}{\lambda_{gmn}}$$

$$\lambda_{gmn} = \frac{2\pi}{k_{zmn}} = \frac{2\pi}{\sqrt{k^2 - k_{cmn}^2}} = \frac{2\pi}{k\sqrt{1-\left(\frac{k_{cmn}}{k}\right)^2}} = \frac{\lambda}{\sqrt{1-\left(\frac{\lambda}{\lambda_{cmn}}\right)^2}} = \frac{\lambda}{\sqrt{1-\left(\frac{f_{cmn}}{f}\right)^2}}$$

Phase velocity:

$$v_{pmn} = \frac{\omega}{k_{zmn}} = \frac{\omega}{\sqrt{k^2 - k_{cmn}^2}} = \frac{c}{\sqrt{1-\left(\frac{\lambda}{\lambda_{cmn}}\right)^2}} > c$$

Energy velocity:

$$v_{emn} = \frac{\overline{P}_z}{\overline{W}_e + \overline{W}_m} = c\sqrt{1-\left(\frac{\lambda}{\lambda_{cmn}}\right)^2}$$

Rectangular Waveguide (13)

Group velocity:

$$v_{gmn} = \frac{1}{\frac{dk_{zmn}}{d\omega}} = \frac{1}{\frac{d\sqrt{k^2 - k_{cmn}^2}}{d\omega}} = \frac{1}{\frac{k}{\sqrt{k^2 - k_{cmn}^2}} \frac{dk}{d\omega}} = \frac{1}{\frac{1}{\sqrt{1-\left(\frac{k_{cmn}}{k}\right)^2}} \sqrt{\mu\varepsilon}}$$

$$= c\sqrt{1-\left(\frac{k_{cmn}}{k}\right)^2} = c\sqrt{1-\left(\frac{\lambda}{\lambda_{cmn}}\right)^2} < c$$

Rectangular Waveguide (14)

Wave impedance:

$$Z_w = \frac{E_x}{H_y} = -\frac{E_y}{H_x}$$

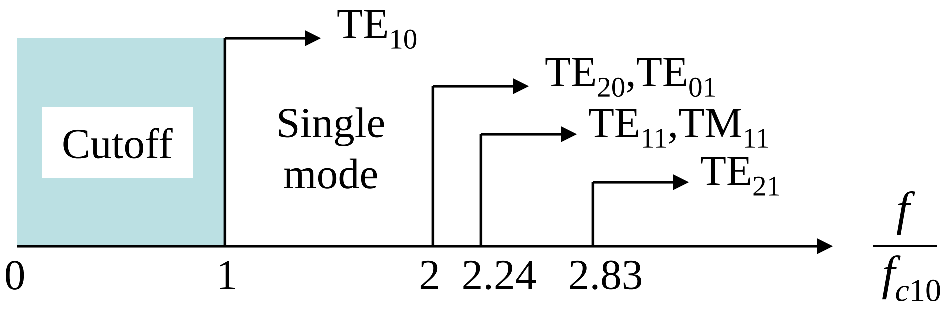
$$Z_{wmn}^{\text{TE}} = \frac{\eta}{\sqrt{1-\left(\frac{\lambda}{\lambda_{cmn}}\right)^2}} = \frac{\eta}{\sqrt{1-\left(\frac{f_{cmn}}{f}\right)^2}} = \begin{cases} \text{Real} & f > f_c \quad \text{or} \quad \lambda > \lambda_c \\ \infty & f = f_c \quad \text{or} \quad \lambda = \lambda_c \\ \text{Imaginary} & f < f_c \quad \text{or} \quad \lambda < \lambda_c \end{cases}$$

$$Z_{wmn}^{\text{TM}} = \eta\sqrt{1-\left(\frac{\lambda}{\lambda_{cmn}}\right)^2} = \eta\sqrt{1-\left(\frac{f_{cmn}}{f}\right)^2} = \begin{cases} \text{Real} & f > f_c \quad \text{or} \quad \lambda > \lambda_c \\ \infty & f = f_c \quad \text{or} \quad \lambda = \lambda_c \\ \text{Imaginary} & f < f_c \quad \text{or} \quad \lambda < \lambda_c \end{cases}$$

Rectangular Waveguide (15)

Consider $f_{cmn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$ $m,n = 0,1,2,\ldots$

except for $m = n = 0$



Define the dominant mode as the mode having the lowest cutoff frequency. Assume $a > b$, the dominant mode: TE_{z10}

$$m = 1,\ \ n = 0$$

$$f_{c10} = \frac{1}{2a\sqrt{\mu\varepsilon}} \qquad k_{c10} = \frac{\pi}{a} \qquad \lambda_{c10} = 2a$$

Rectangular Waveguide (16)

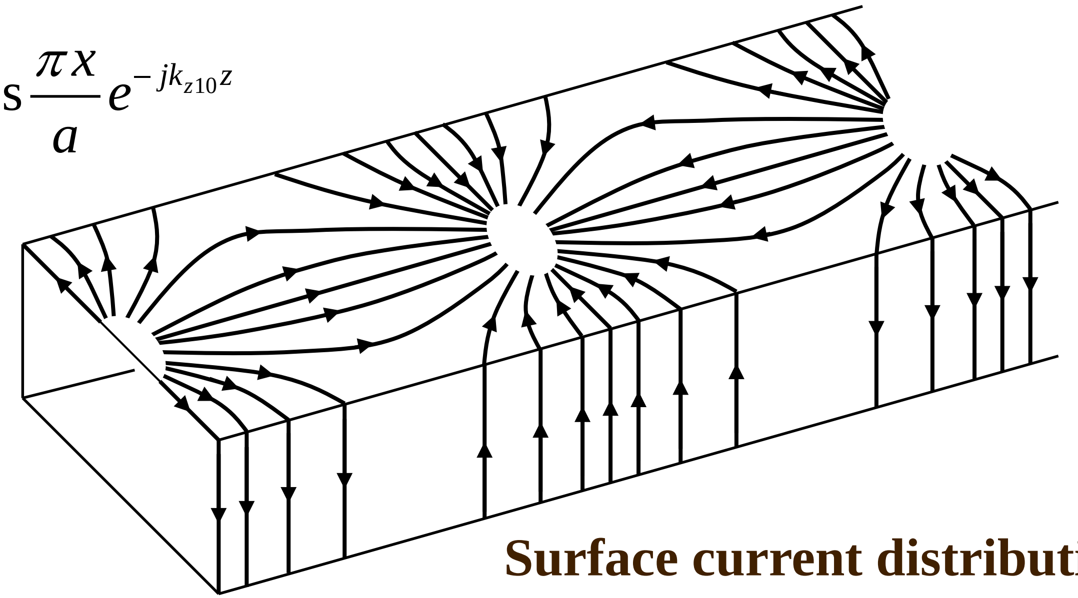
TE_{z10} **modal field:**

$$E_y = -H_{10}\frac{j\omega\mu a}{\pi}\sin\frac{\pi x}{a}e^{-jk_{z10}z}$$

$$H_x = H_{10}\frac{jk_{z10}a}{\pi}\sin\frac{\pi x}{a}e^{-jk_{z10}z}$$

$$H_z = H_{10}\cos\frac{\pi x}{a}e^{-jk_{z10}z}$$

$$\mathbf{J}_s = \hat{n}\times\mathbf{H}$$



Surface current distribution

Rectangular Waveguide (17)

Express the modal field in terms of plane waves:

$$\mathbf{E} = -\hat{y}\frac{k\eta}{k_{x10}}\frac{H_{10}}{2}\left[e^{j(k_{x10}x - k_{z10}z)} - e^{-j(k_{x10}x + k_{z10}z)}\right]$$

Superposition of two plane waves!