Coulomb’s Law (1)

$$E = \hat{R} \frac{q}{4\pi \varepsilon R^2} \quad (\text{V/m})$$

linear superposition

electric dipole
Equation of electrostatics

- **Electrostatics**
  - $\partial/\partial t = \omega = 0 \Rightarrow$ two uncoupled sets of (static) equations are obtained for $E$ & $D$ and $H$ & $B$
  - Equation of electrostatics
    - $\nabla \times \mathbf{E} = 0$ or $\int_{C} \mathbf{E} \cdot d\mathbf{l} = 0$
    - $\nabla \cdot \mathbf{D} = \rho$
      - $\oint_{S} \mathbf{D} \cdot d\mathbf{s} = Q$
  - Boundary conditions
    - $\mathbf{n}_{12} \times (\mathbf{E}_{2} - \mathbf{E}_{1}) = 0$
    - $\mathbf{n}_{12} \cdot (\mathbf{D}_{2} - \mathbf{D}_{1}) = \rho_{s}$
  - Constitutive relation $\mathbf{D} = \varepsilon \mathbf{E}$
Coulomb’s Law

- Electric field due to a single charge
  \[ E = \hat{R} \frac{q}{4\pi \varepsilon R^2} \]
- Electric field due to many discrete charges
  \[ E = \frac{1}{4\pi \varepsilon} \sum_{i=1}^{N} \frac{q_i (\mathbf{R} - \mathbf{R}_i)}{|\mathbf{R} - \mathbf{R}_i|^3} \]
- Electric field due to charge distributions

\[ dE = \hat{R}' \frac{dq}{\Delta \varepsilon R'^2} = \frac{\hat{R}'}{4\pi \varepsilon R'^2} \]
\[ E = \int_{V'} dE = \frac{1}{4\pi \varepsilon} \int_{V'} \frac{\hat{R}'}{R'^2} \]
(volume distribution).
\[ E = \frac{1}{4\pi \varepsilon} \int_{S'} \frac{\hat{R}'}{R'^2} \]
(surface distribution).
\[ E = \frac{1}{4\pi \varepsilon} \int_{L'} \frac{\hat{R}'}{R'^2} \]
(line distribution).

Figure 4-4: The electric field \( E \) at \( P \) due to two charges is equal to the vector sum of \( E_1 \) and \( E_2 \).
Coulomb’s Law (example 1)

- Field due to a ring of charge $\rho_l$

\[
dq = \rho_l \, dl = \rho_l b \, d\phi
\]

\[
\mathbf{R}_1' = -\hat{b} \hat{\phi} + \hat{h} h,
\]

\[
R_1' = |\mathbf{R}_1'| = \sqrt{b^2 + h^2}, \quad \mathbf{\hat{R}}_1' = \frac{\mathbf{R}_1'}{|\mathbf{R}_1'|} = \frac{-\hat{b} \hat{\phi} + \hat{h} h}{\sqrt{b^2 + h^2}}
\]

\[
dE_1 = \frac{1}{4\pi \varepsilon_0} \frac{\mathbf{\hat{R}}_1' \rho_l \, dl}{\mathbf{R}_1'^2} = \frac{\rho_l b}{4\pi \varepsilon_0} \frac{(-\hat{b} \hat{\phi} + \hat{h} h)}{(b^2 + h^2)^{3/2}} \, d\phi
\]

\[
dE = dE_1 + dE_2 = \hat{z} \frac{\rho_l b h}{2\pi \varepsilon_0} \frac{d\phi}{(b^2 + h^2)^{3/2}}
\]

\[
E = \hat{z} \frac{\rho_l b h}{2\pi \varepsilon_0 (b^2 + h^2)^{3/2}} \int_0^{\pi} d\phi
\]

\[
= \hat{z} \frac{\rho_l b h}{2\varepsilon_0 (b^2 + h^2)^{3/2}}
\]

\[
= \hat{z} \frac{h}{4\pi \varepsilon_0 (b^2 + h^2)^{3/2}} Q,
\]
Coulomb’s Law (example 2)

- Field due to a disk of charge $\rho_s$

$$dq = \rho_s \, ds = 2\pi \rho_s r \, dr.$$  

$$dE = \hat{z} \frac{h}{4\pi \varepsilon_0 (r^2 + h^2)^{3/2}} (2\pi \rho_s r \, dr)$$

$$E = \hat{z} \frac{\rho_s h}{2\varepsilon_0} \int_0^a \frac{r \, dr}{(r^2 + h^2)^{3/2}}$$

$$= \pm \hat{z} \frac{\rho_s}{2\varepsilon_0} \left[ 1 - \frac{|h|}{\sqrt{a^2 + h^2}} \right]$$

$$E = \hat{z} \frac{\rho_s}{2\varepsilon_0} \quad \text{(infinite sheet of charge).}$$

field due to a plane of charge is a constant
Gauss’ Law

- Differential form
  \[ \nabla \cdot D = \rho \]
  \[ \oint_S D \cdot ds = Q \]

- Integral form

- Field due to a sphere or point charge

\[
D(R) = \hat{R}D_R,
\]
\[
\oint_S D \cdot ds = \oint_S \hat{R}D_R \cdot \hat{R} ds
\]
\[
= \oint_S D_R ds = D_R(4\pi R^2) = q.
\]

\[
E(R) = \frac{D(R)}{\varepsilon} = \hat{R} \frac{q}{4\pi \varepsilon R^2}
\]

Coulomb's law
Gauss’ Law (example)

- Field due to an infinite line charge $\rho_l$

\[
D = \hat{r} D_r.
\]

\[
\mathcal{Q} = \rho_l h.
\]

\[
\int_{z=0}^{h} \int_{\phi=0}^{2\pi} \hat{r} D_r \cdot \hat{r} r \, d\phi \, dz = \rho_l h
\]

\[
2\pi h D_r r = \rho_l h.
\]

\[
E = \frac{D}{\varepsilon_0} = \hat{r} \frac{D_r}{\varepsilon_0} = \hat{r} \frac{\rho_l}{2\pi \varepsilon_0 r} \quad \text{(infinite line of charge)}.
\]
Electric scalar potential (1)

- Potential and electric field
  - Electric force applied to a charge in a constant field \( F_e = qE \)
  - External force to move the charge with a constant speed \( F_{ext} = -F_e = -qE \)
  - Work/energy to move the charge over the distance \( dW = F_{ext} \cdot dl = -qE \cdot dl \) (J).

- Electric potential \( dV = \frac{dW}{q} = -E \cdot dl \) (J/C or V).

- Potential difference (voltage)
  \[ V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} E \cdot dl \]
Electric scalar potential (2)

- Properties of the potential
  - No dependence on the path
    \[ V_{21} = V_2 - V_1 = - \int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}, \]
  - Closed line integral vanishes
    \[ \oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \quad \text{(Electrostatics).} \]
  - Electrostatic field is conservative (potential) \( \nabla \times \mathbf{E} = 0. \)
  - Kirchhoff’s voltage law is a particular case: The voltage drop is zero in a closed loop
  - Potential at a point in space is always defined with respect to some other point in space, e.g. \( \infty \)
    \[ V = - \int_{\infty}^{P} \mathbf{E} \cdot d\mathbf{l} \quad \text{(V)} \]
Electric scalar potential (3)

- Potential due to various charge distributions
  - Point charge
    \[ d\mathbf{l} = \hat{\mathbf{r}} \, dR \quad V = - \int_{R}^{\infty} \left( \hat{\mathbf{r}} \frac{q}{4\pi \varepsilon R^2} \right) \cdot \hat{\mathbf{r}} \, dR = \frac{q}{4\pi \varepsilon R} \]
    \[ V(R) = \frac{q}{4\pi \varepsilon |R - R_1|} \]
  
  - Many discrete charges
    \[ V(R) = \frac{1}{4\pi \varepsilon} \sum_{i=1}^{N} \frac{q_i}{|R - R_i|} \]
  
  - Various continuous charge distributions in free space
    \[ V(R) = \frac{1}{4\pi \varepsilon} \int_{V'} \frac{\rho_v}{R'} \, dV' \quad \text{(volume distribution)}, \quad R' = |R - R_i| \]
    \[ V(R) = \frac{1}{4\pi \varepsilon} \int_{S'} \frac{\rho_s}{R'} \, ds' \quad \text{(surface distribution)}, \]
    \[ V(R) = \frac{1}{4\pi \varepsilon} \int_{L'} \frac{\rho_l}{R'} \, dl' \quad \text{(line distribution)}. \]
Electric scalar potential (4)

- Electric field through electric potential
  - Differential of potential
    - Through electric field
      \[ dV = -E \cdot dl. \]
    - Through potential (directional derivative)
      \[ dV = \nabla V \cdot dl, \]
  - Electric field and potential
    \[ E = -\nabla V \]
- What is potential?
  - Has physical meaning of voltage
  - Can be understood as an auxiliary function, which helps find the electric field
  - The idea of “potentials” as auxiliary functions can be extended to more complicated potentials
Electric scalar potential (5)

- Electric field due to a dipole
  - Electric dipole represents two charges of opposite polarity separated by a small distance
  - Potential due to a dipole

\[
V = \frac{1}{4\pi \varepsilon_0} \left( \frac{q}{R_1} + \frac{-q}{R_2} \right) = \frac{q}{4\pi \varepsilon_0} \left( \frac{R_2 - R_1}{R_1 R_2} \right)
\]

\[d \ll R \Rightarrow R_2 - R_1 \simeq d \cos \theta, \quad R_1 R_2 \simeq R^2.
\]

\[p\text{-dipole moment}
\]

\[\Rightarrow V = \frac{qd}{4\pi \varepsilon_0 R^2} \cos \theta = \frac{\mathbf{p} \cdot \hat{\mathbf{R}}}{4\pi \varepsilon_0 R^2}
\]

- Electric field due to a dipole

\[
\mathbf{E} = \frac{qd}{4\pi \varepsilon_0 R^3} \left( \hat{\mathbf{R}} \cdot 2 \cos \theta + \hat{\mathbf{\theta}} \cdot \sin \theta \right)
\]
Differential equations for the potential

- **Poisson equation**
  - Use Gauss law: \( \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon} \implies \nabla \cdot (\nabla V) = -\frac{\rho}{\varepsilon} \)
  - Laplacian: \( \nabla^2 V = \nabla \cdot (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \)
  - Poisson equation: \( \nabla^2 V = -\frac{\rho}{\varepsilon} \)
  - Solution for free space: \( V = \frac{1}{4\pi \varepsilon} \int_{V'} \frac{\rho_{v'}}{R'} \, dV' \)

- **Laplace equation (no charges)**: \( \nabla^2 V = 0 \)
Properties of materials (1)

• Conductors
  — Drift velocity $u_e = -\mu_e E$ (m/s),
  — Conduction current density (point Ohm’s law)
    \[ J = \sigma E \] (A/m$^2$)
  — Conductivity $\sigma = -\rho_e \mu_e = N_e \mu_e e$ (S/m)
    Example: copper $\sigma = 5.8 \times 10^7$ S/m
  — Perfect electric conductor (PEC)
    \[ J = 0 \]
    \[ E = 0 \]
Properties of materials (2)

- **Dielectrics**

  - Electric flux density \( \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \)
  
  - \( \mathbf{P} \) - electric polarization field or polarization density (means electric dipole per unit volume)
  
  - For isotropic linear homogeneous materials \( \mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \), \( \chi_e \) – polarisability
  
  - Permittivity \( \mathbf{D} = \varepsilon_0 (1 + \chi_e) \mathbf{E} = \varepsilon \mathbf{E} \Rightarrow \varepsilon = \varepsilon_0 (1 + \chi_e) \)
Properties of materials (3)

- **Resistance and conductance**
  - Consider a linear resistor
  - **Voltage**
    \[ V = V_1 - V_2 = -\int_{x_2}^{x_1} E \cdot dl = -\int_{x_2}^{x_1} \hat{\mathbf{E}} \cdot \hat{\mathbf{d}}l = E_x l \]
  - **Current**
    \[ I = \int_A \mathbf{J} \cdot ds = \int_A \sigma \mathbf{E} \cdot ds = \sigma E_x A \]
  - **Resistance**
    \[ R = \frac{l}{\sigma A} \quad (\Omega) \]
  - **Conductance**
    \[ G = \frac{1}{R} \]
  - **Generalized expression**
    \[ R = \frac{V}{I} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \mathbf{J} \cdot ds} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot ds} \]
Properties of materials (4)

• **Conductance of a coaxial cable**
  
  - Consider a coaxial resistor
  - Current density
    \[ J = \hat{r} \frac{I}{A} = \hat{r} \frac{I}{2\pi rl} \]
  
  - Electric field
    \[ J = \sigma E \quad \Rightarrow \quad E = \hat{r} \frac{I}{2\pi \sigma rl} \]
  
  - Voltage
    \[ V_{ab} = -\int_{b}^{a} E \cdot dl = -\int_{b}^{a} \frac{I}{2\pi \sigma l} \hat{r} \cdot \hat{r} \, dr \]
    \[ = \frac{I}{2\pi \sigma l} \ln \left( \frac{b}{a} \right). \]
  
  - Conductance per unit length (as was defined for TLs)
    \[ G' = \frac{G}{l} = \frac{1}{Rl} = \frac{I}{V_{abl}} = \frac{2\pi \sigma}{\ln(b/a)} \quad (S/m) \]
Properties of materials (5)

- **Capacitance**
  - Consider a capacitor consisting of two conductors with a voltage
  - Opposite charges appear on the conductors
  - Capacitance \[ C = \frac{Q}{V} \text{ (C/V or F)} \]
  - Alternative expression

\[
Q = \int_S \rho_s \, ds = \int_S \varepsilon \hat{n} \cdot E \, ds = \int_S \varepsilon E \cdot ds,
\]

\[
V = V_{12} = -\int_{P_2}^{P_1} E \cdot dl,
\]

- Relation to resistance \[ RC = \frac{\varepsilon}{\sigma} \]
Properties of materials (6)

- Special case of capacitors
  - Parallel plate capacitor
    \[ E = -\hat{z}E, \]
    \[ V = -\int_{0}^{d} E \cdot dl = -\int_{0}^{d} (-\hat{z}E) \cdot \hat{z} dz = Ed, \]
    \[ C = \frac{Q}{V} = \frac{Q}{Ed} = \frac{\varepsilon A}{d} \]
  
  - Coaxial cable
    \[ RC = \frac{\varepsilon}{\sigma} \]
    \[ C = \frac{Q}{V} = \frac{2\pi \varepsilon l}{\ln(b/a)} \quad C' = \frac{C}{l} = \frac{2\pi \varepsilon}{\ln(b/a)} \]
Numerical capacitance extraction (1)

- Integral equation method
  - Given
    - Two conductors
    - Voltage on the conductors is $V_0/2$ and $-V_0/2$
    - Find the capacitance $C$
  - Formulation
    - Potential on the surfaces is given by $V_0/2$ and $-V_0/2$
    - Potential everywhere is calculated as $V(r) = \int \frac{1}{4\pi \varepsilon_0 |r - r'|} \rho_s(r') \, dS'$
      where $\rho_s$ is an unknown surface charge
    - Equate the potentials on the surfaces

$$\implies \int \int_{S^{(1)} + S^{(2)}} \frac{1}{4\pi \varepsilon_0 |r - r'|} \rho_s(r') \, dS' = \begin{cases} V_0/2; & r \in S_1 \\ -V_0/2; & r \in S_2 \end{cases}$$
Numerical capacitance extraction (2)

- **Matrix equation**
  - **Discretize the problem**
    - Divide the surfaces into \( N = N_{S^{(1)}} + N_{S^{(2)}} \) small patches, \( N_{S^{(1)}}, N_{S^{(2)}} \) are numbers of patches on surfaces \( S^{(1)}, S^{(2)} \)
    - Replace the integration by summation
    - Enforce the IE at the location of charges
  - \[
  \Rightarrow \sum_{n=1}^{N} \iint_{S_n} \frac{\rho_s(r') ds'}{4\pi\epsilon_0 |r_m - r'|} = \begin{cases} 
  V_0/2; & r_m \in S_1 \\
  -V_0/2; & r_m \in S_2
  \end{cases}
  \]
  - **Obtain a matrix equation**
  \[
  \sum_{n=1}^{N} \iint_{S_n} \frac{\rho_s(r') ds'}{4\pi\epsilon_0 |r_m - r'|} = \begin{cases} 
  V_0/2; & r_m \in S_1 \\
  -V_0/2; & r_m \in S_2
  \end{cases} \Rightarrow \sum_{n=1}^{N} Z_{mn} Q_n = V_m \Rightarrow ZQ = V
  \]
Numerical capacitance extraction (3)

- **Matrix equation and solution**
  - **Matrix equation**
    \[ ZQ = V \]
  
  \[ Z = N \times N \text{ matrix}; \quad Z_{mn} = \frac{1}{\Delta S_n} \int \int_{S_n} \frac{ds'}{4\pi \epsilon_0 |r_m - r'|} \Rightarrow \]

  \[ \begin{cases} 
  Z_{mn} \approx \frac{1}{4\pi \epsilon_0 |r_m - r'|} ; \quad r_m \neq r_n \\
  Z_{mn} \approx \frac{1}{2\epsilon_0 \sqrt{\pi \Delta S_n}} ; \quad r_m = r_n 
  \end{cases} \]

  \[ Q = N \text{ vector}; \quad Q_n = \rho_{s}(r_n) \Delta S_n \]

  \[ V = N \text{ vector}; \quad V_m = \begin{cases} 
  V_0/2 ; \quad r_m \in S_1 \\
  -V_0/2 ; \quad r_m \in S_2 
  \end{cases} \]

  - **Solution**
    \[ Q = Z^{-1}V \]

- **Extracted capacitance**
  \[ C = \frac{Q_{S^{(1)}}}{V_0} \Rightarrow C = \frac{\sum_{n=1}^{N_{S^{(1)}}} Q_n}{V_0} \]
Image theory

- Consider a charge above a PEC
  - How can we find the field?
    - Coulomb’s law (superposition) is hard to apply
    - Numerical methods require work
    - For simple structures use images!
  - Images are auxiliary objects
  - Images are chosen so that the boundary conditions on the PEC are satisfied, i.e. \( E_{\text{tan}} = 0 \Rightarrow V = \text{const} \)
  - For planar PECs the image is an opposite charge at a distance \(-d\) from the PEC
  - More complicated images can be used for other simple configurations: e.g. spheres and cylinders
Electric energy

- Consider a capacitor
  - The amount of energy to transfer a differential charge \( dq \)
    \[
    dW_e = \frac{q}{C} dq
    \]
  - For a total charge \( Q \)
    \[
    W_e = \int_0^Q \frac{q}{C} dq = \frac{1}{2} \frac{Q^2}{C}
    \]
    \[
    \Rightarrow W_e = \frac{1}{2} CV^2 \quad (J)
    \]
  - For a parallel plate capacitor \( C = \varepsilon A/d \), \( V = Ed \)
    \[
    \Rightarrow W_e = \frac{1}{2} \frac{\varepsilon A}{d} (Ed)^2 = \frac{1}{2} \varepsilon E^2 (Ad) = \frac{1}{2} \varepsilon E^2 V
    \]
  - Electric energy density (applies to ANY structure)
    \[
    w_e = \frac{W_e}{V} = \frac{1}{2} \varepsilon E^2 \quad (J/m^3)
    \]
  - Electric energy in a volume
    \[
    W_e = \frac{1}{2} \int_V \varepsilon E^2 dV \quad (J)
    \]
Fields and geometries

One conductor is made up of two spheres connected by a wire.

Ignore interaction between spheres on charge distribution

\[ V(\vec{R}) = \frac{1}{4\pi\varepsilon} \int_{s}^{s'} \frac{\rho_s}{R'} ds' \]

at surface of spheres

\[ V_a = \frac{1}{4\pi\varepsilon} \frac{Q}{a} \quad \text{and} \quad V_b = \frac{1}{4\pi\varepsilon} \frac{q}{b} \]

\[ V_b = V_a \quad \rightarrow \quad \frac{Q}{a} = \frac{q}{b} \]

\[ E_a = \frac{Q}{4\pi\varepsilon a^2} \quad \text{and} \quad E_b = \frac{q}{4\pi\varepsilon b^2} \]

\[ \frac{E_a}{E_b} = \frac{Q / a^2}{q / b^2} = \frac{b}{a} \quad \rightarrow \quad E_b = \frac{a}{b} E_a \]
High electric fields at sharp points and defects. (equivalent to small radius feature). Leads to breakdown and failure.

Source of electrons (e.g. electron microscope or field emission microscope)
Electron sources
Field ion microscope (FIM)

Works best for refractory metals $W$, $Mo$, $Pt$, $Ir$ etc.

Rezeg *et al.*, Journal of Chemical Physics, 28 May 2006
Field ion microscope (FIM)

Imaging at 5.0 kV, Manipulating at 6.0 kV

Single atom on tungsten tip
Imaged at 2.1 kV
Used for SPM
He ion microscope
Electron source

A. Schirmeisen et al.

Figure 1. $V(x)$ is potential energy as a function of distance away from the surface of a specimen, $x = 0$. $V_a(x)$ is the potential energy of an atom in the absence of an electric field, $E$, and $V_i(x)$ is the potential energy of an ion in its $i$th ionization state in the presence of $E$. $l_i$ is the $i$th ionization state of an ion, $e$ is the charge on an electron, $n$ is the number of electronic charges, $\Lambda$ is the cohesive energy of a solid, $\phi_0$ is the local work function, $Q_a(E)$ is the height of the so-called Schottky hump, and $x_a$ is the position of the maximum value of the Schottky hump.