ECE 107: Electromagnetism

Set 7: Dynamic fields

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Maxwell’s equations

- Maxwell’s equations
  - \[ \nabla \times E = -\frac{\partial B}{\partial t} \]
  - \[ \nabla \times H = \frac{\partial D}{\partial t} + J \]
  - \[ \nabla \cdot D = \rho \]
  - \[ \nabla \cdot B = 0 \]

\[ \hat{n}_{12} \times (E_2 - E_1) = 0 \]
\[ \hat{n}_{12} \cdot (D_2 - D_1) = \rho_s \]
\[ \hat{n}_{12} \times (H_2 - H_1) = J_s \]
\[ \hat{n}_{12} \cdot (B_2 - B_1) = 0 \]

- Goals of this chapter
  - To “close the loop” between dynamics and statics
  - To explain effects introduced by time variations
  - To extend some results from static to dynamics

\[ \oint E \cdot dl = -\frac{d}{dt} \iint B \cdot ds \]
\[ \oint H \cdot dl = \frac{d}{dt} \iint D \cdot ds + I \]
\[ \iint D \cdot ds = Q \]
\[ \iint B \cdot ds = 0 \]
\[ D = \varepsilon E \]
\[ B = \mu H \]
Faraday’s law

- Faraday’s law: \[ \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} \]
- Magnetic flux: \[ \Phi = \int_S \mathbf{B} \cdot d\mathbf{s} \]
- Electromotive force voltage: \[ V_{\text{emf}} = -\frac{d\Phi}{dt} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} \]
- \( V_{\text{emf}} \) induces currents in a wire loop. The currents produce the flux that opposes the inducing flux (Lenz’s law)
- \( V_{\text{emf}} \) allows for transformers, generators, motors, etc. Its existence is a foundation of Electromagnetics
Faraday’s law (2)

- Transformers:
  - Current $\rightarrow$ mutual flux
  - Flux $\rightarrow$ voltage

$$V_1 = -N_1 \frac{d\Phi}{dt}, \quad V_2 = -N_2 \frac{d\Phi}{dt}$$

$$\Rightarrow \frac{V_1}{V_2} = \frac{N_1}{N_2}$$

- Energy conservation $\rightarrow$

$$I_1 V_1 = I_2 V_2$$

$$\Rightarrow \frac{I_1}{I_2} = \frac{N_2}{N_1}$$

Figure 6-5: In a transformer, the directions of $I_1$ and $I_2$ are such that the flux $\Phi$ generated by one of them is opposite that generated by the other. The direction of the secondary winding in (b) is opposite that in (a), and so are the direction of $I_2$ and the polarity of $V_2$. 
Faraday’s law (2)

- **Electric generator:**
  - Rotating loop in a magnetic field $\rightarrow$ EMF
  
  \[
  V_{\text{emf}} = -\frac{d\Phi}{dt} = -\frac{d}{dt}[B_0 A \cos(\omega t + C_0)]
  = A\omega B_0 \sin(\omega t + C_0),
  \]

- **Electric motor:**
  - Opposite to the generator
  - An AC current is sent to the coil in a constant magnetic field $\rightarrow$ coil spins

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Faraday’s law (2)

- **Eddy currents:**
  - Time varying magnetic field $\rightarrow$ Electric field $\rightarrow$ (Eddy) current
  \[ \nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \quad \Rightarrow \quad \mathbf{J} = \sigma \mathbf{E} \]

- **Examples:**
  - Eddy current waist separator
  - Eddy current break
  - Transcranial magnetic stimulation
Ampere’s law

- **Ampere’s law**: \[ \oint_{c} \mathbf{H} \cdot d\mathbf{l} = I + \frac{d}{dt} \iint_{S} \mathbf{D} \cdot ds \]
- **Displacement current**: \[ I_d = \iint_{S} J_d \cdot ds = \frac{d}{dt} \iint_{S} \mathbf{D} \cdot ds \]
- **Displacement current density**: \[ J_d = \frac{d\mathbf{D}}{dt} \]
- **Displacement current** is required to establish consistent equations of Electromagnetics. Its introduction resulted in Maxwell’s equations.
- To explain its importance, consider a capacitor under an ac voltage.

![Figure 6-13](image_url) The displacement current \( I_{24} \) in the insulating material of the capacitor is equal to the conducting current \( I_{1c} \) in the wire.
Charge-current continuity relation

• Start with the Ampere’s law \( \nabla \times H = \frac{\partial D}{\partial t} + J \)

• Apply divergence \( \nabla \cdot (\nabla \times H) = \nabla \cdot \frac{\partial D}{\partial t} + \nabla \cdot J \Rightarrow \nabla \cdot J = -\frac{\partial}{\partial t} \nabla \cdot D \)

  (vector identity)

• Continuity relation: \( \Rightarrow \nabla \cdot J_v = -\frac{\partial}{\partial t} \rho_v \)

  \( \Rightarrow \iiint_S J_v \cdot ds = -\frac{\partial}{\partial t} \iiint_V \rho_v \Rightarrow I = -\frac{\partial}{\partial t} Q \)

• The amount of charge change equals the current needed to compensate the change

• For statics
  – The current and charge are independent
  – \( \iiint_S J_v \cdot ds = 0 \Rightarrow \text{Kirchhoff’s current law} \sum_i I_i = 0 \)
Complex permittivity (1)

- Modified Ampere’s law
  - Ampere’s law \( \nabla \times H = j \omega D + J \)
  - Constitutive relation \( D = \varepsilon E \)
  - Current \( J = J_c + J_i \)
    - Conductivity \( J_c = \sigma E \)
    - Impressed given source
  - Modified Ampere’s law
    \[ \nabla \times H = j \omega \varepsilon E + J_c + J_i \]
    \[ \nabla \times H = j \omega \varepsilon E + \sigma E + J_i \]
    \[ \nabla \times H = j \omega \left( \varepsilon - \frac{j\sigma}{\omega} \right) E + J_i \]

- Complex permittivity \( \varepsilon_c = \varepsilon - \frac{j\sigma}{\omega} = \varepsilon' - j\varepsilon'' = |\varepsilon_c| e^{-j\delta} \)
- Loss tangent
  \[ \tan \delta = \varepsilon'' / \varepsilon' = \sigma / (\omega \varepsilon) \]
Complex permittivity (2)

- Maxwell’s equations in conducting media
  - Differential equations
    \[ \nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \]
    \[ \nabla \times \mathbf{H} = j\omega\varepsilon_{\text{c}}\mathbf{E} + \mathbf{J}_i \]
    \[ \nabla \cdot \mathbf{D} = \rho_i \]
    \[ \nabla \cdot \mathbf{B} = 0 \]
  
  - Boundary conditions
    \[ \hat{n}_{12} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 \]
    \[ \hat{n}_{12} \cdot (\varepsilon_{c2}\mathbf{E}_2 - \varepsilon_{c1}\mathbf{E}_1) = \rho_{is} \]
    \[ \hat{n}_{12} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_{is} \]
    \[ \hat{n}_{12} \cdot (\mu_2\mathbf{H}_2 - \mu_1\mathbf{H}_1) = 0 \]
  
  - Continuity relation
    \[ \nabla \cdot \mathbf{J}_i = -j\omega\rho_{is} \]
Electromagnetic potentials (1)

- Fields radiated in free space by a charge distribution $\rho_v(r,t)$ and current distribution $J_v(r,t)$ (recall that $\rho_v$ and $J_v$ are related $\nabla \cdot J_v = -\frac{\partial \rho_v}{\partial t}$).

\[
H = \frac{1}{\mu} \nabla \times A
\]

\[
E = -\nabla V - \frac{\partial A}{\partial t}
\]

Retarded potential
\[
V(R,t) = \frac{1}{4\pi \varepsilon} \iiint_{V'} \rho_v \left( t - R'/v_p \right) \frac{dv'}{R'}
\]

\[
A(R,t) = \frac{\mu}{4\pi} \iiint_{V'} J_v \left( t - R'/v_p \right) \frac{dv'}{R'}
\]

\[v_p = c = 3 \times 10^8 \text{ m/s}\]
Electromagnetic potentials (2)

- Frequency domain fields radiated by a charge distribution \( \tilde{\rho}_v(\mathbf{r}) \) and current distribution (recall that \( \tilde{\rho}_v \) and \( \tilde{\mathbf{J}}_v \) are related \( \nabla \cdot \tilde{\mathbf{J}}_v = -j\omega \tilde{\rho}_v \) )

\[
\rho_v(R_i, t - R'/u_p) = \Re \left[ \tilde{\rho}_v(R_i) e^{ij\omega(t - R'/u_p)} \right] = \Re \left[ \tilde{\rho}_v(R_i) e^{-j\omega R'/u_p e^{j\omega t}} \right] = \Re \left[ \tilde{\rho}_v(R_i) e^{-jk R'} e^{j\omega t} \right]
\]

\[
V(R, t) = \Re \left[ \tilde{V}(R) e^{j\omega t} \right] = \Re \left[ \frac{1}{4\pi \epsilon} \int_{V'} \frac{\tilde{\rho}_v(R_i) e^{-jk R'}}{R'} e^{j\omega t} dv' \right]
\]

\[
A(R, t) = \Re \left[ \tilde{A}(R) e^{j\omega t} \right]
\]

\[
k = \frac{\omega}{v_p} = \frac{2\pi}{\lambda} \quad \text{wavenumber}
\]

\[
\tilde{\mathbf{H}} = \frac{1}{\mu} \nabla \times \tilde{\mathbf{A}}
\]

\[
\Rightarrow \quad \tilde{\mathbf{E}} = -\nabla \tilde{V} - j\omega \tilde{\mathbf{A}}
\]

\[
\tilde{V}(R) = \frac{1}{4\pi \epsilon} \iiint_{V'} \tilde{\rho}_v e^{-jk R'} \frac{dv'}{R'}
\]

\[
\tilde{A}(R) = \frac{\mu}{4\pi} \iiint_{V'} \tilde{\mathbf{J}}_v e^{-jk R'} \frac{dv'}{R'}
\]
Radiation from a short dipole (1)

- Consider an electric dipole \( \mathbf{p} = ql \hat{z} \)
- Its time derivative results in current
  \[ \frac{d}{dt} \mathbf{p} = Il \hat{z} \Rightarrow j\omega \tilde{\mathbf{p}} = Il \hat{z} \]
- When \( l = \lambda \) then \( I \) is uniform
- Physically, the dipole is represented by two wires
- Vector potential
  \[ \mathbf{\tilde{A}} = \frac{\mu_0}{4\pi} \frac{e^{-jkR}}{R} \int_{-l/2}^{l/2} \hat{z}I_0 \, dz = \hat{z} \frac{\mu_0}{4\pi} I_0 l \left( \frac{e^{-jkR}}{R} \right) \]
  \[ \hat{z} = \hat{R} \cos \theta - \hat{\phi} \sin \theta. \]
Radiation from a short dipole (2)

- Expressions for the radiated fields

\[ \vec{H} = \frac{1}{\mu_0} \nabla \times \vec{A}, \quad \vec{E} = \frac{1}{j\omega\varepsilon_0} \nabla \times \vec{H}, \]

\[ \vec{H}_\phi = \frac{I_0 l k^2}{4\pi} e^{-jkr} \left[ \frac{j}{kR} + \frac{1}{(kR)^2} \right] \sin \theta \]

\[ \vec{E}_R = \frac{2I_0 l k^2}{4\pi \eta_0} e^{-jkr} \left[ \frac{1}{(kR)^2} - \frac{j}{(kR)^3} \right] \cos \theta \]

\[ \vec{E}_\theta = \frac{I_0 l k^2}{4\pi \eta_0} e^{-jkr} \left[ \frac{j}{kR} + \frac{1}{(kR)^2} - \frac{j}{(kR)^3} \right] \sin \theta \]

- When \( \omega \to 0 \) the expressions reduce to the expression due to a static dipole with \( p = \frac{II}{(j\omega)} \)

- Magnetic field is present only in the dynamic case

- No electric field is in the \( \phi \) direction, hence the field is TM to \( z \)

- The short dipole is one of the simplest antennas
Radiation from a short dipole (3)

- **Far field approximation:** \( R?\lambda \text{ or } kR?1 \)
  - The terms ~\(1/(kR)^2\),\(1/(kR)^3\) can be neglected
  - Far field approximation fields

\[
\tilde{E}_\theta = \frac{jI_0k\eta_0}{4\pi} \left( \frac{e^{-jkR}}{R} \right) \sin \theta \\
\tilde{H}_\phi = \frac{\tilde{E}_\theta}{\eta_0}
\]

- Characteristics of far field fields of ANY antenna
  - Relation between \(E\) and \(H\) is simply through the impedance of free space similar to the case of transmission lines
  - The phase behavior is like for TLs with \(z\) replaced by \(R\)
  - The field decays but only as \(1/R\) (compare to \(1/R^2\) decay for a static charge and \(1/R^3\) decay for an electric dipole)