ECE 107: Electromagnetism

Set 8: Plane waves

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Wave equation

- Source-free lossless Maxwell’s equations
  \[ \nabla \times \tilde{E} = -j \omega \mu \tilde{H} \quad \nabla \cdot \varepsilon \tilde{E} = 0 \]
  \[ \nabla \times \tilde{H} = j \omega \varepsilon \tilde{E} \quad \nabla \cdot \mu \tilde{H} = 0 \]

- Apply curl
  \[ \nabla \times \nabla \times \tilde{E} = -j \omega \mu \nabla \times \tilde{H} \Rightarrow \nabla^2 \tilde{E} + \omega^2 \mu \varepsilon \tilde{E} = 0 \]
  \[ \nabla (\nabla \cdot \tilde{E}) - \nabla^2 \tilde{E} = 0 \]

- Helmholtz equation (HE)
  \[ \nabla^2 \tilde{E} + k^2 \tilde{E} = 0 \]
  \[ \nabla^2 \tilde{H} + k^2 \tilde{H} = 0 \]
  \[ k^2 = \omega^2 \mu \varepsilon \quad \text{wavenumber} \]

- In Cartesian coordinates
  \[ (\nabla^2 + k^2) \tilde{E}_{x,y,z} = 0 \]
Plane waves in free space (1)

- **Radiation from a uniform surface current** $\mathbf{J}_s \mid_{z=0} = \hat{x} J_{s0}$
  
  \[ \Rightarrow \partial/\partial x = \partial/\partial y = 0 \Rightarrow \tilde{\mathbf{E}} = \tilde{E}_x \hat{x}, \quad \tilde{\mathbf{H}} = \tilde{H}_y \hat{y} \]

  - Maxwell’s eqs. reduce to TL equations!
    \[ -\frac{d\tilde{E}_x}{dz} = j\omega\mu\tilde{H}_y \]
    \[ -\frac{d\tilde{H}_y}{dz} = j\omega\varepsilon\tilde{E}_x \]

  - Helmholtz equations reduce to
    \[ \frac{d^2\tilde{E}_x}{dz^2} + k^2 \tilde{E}_x = 0, \quad \frac{d^2\tilde{H}_y}{dz^2} + k^2 \tilde{H}_y = 0 \]

  - Solutions
    \[ \tilde{E}_x = \tilde{E}_{x0} e^{-jkz}, \quad \tilde{H}_y = \frac{\tilde{E}_{x0}^+}{\eta} e^{-jkz} \]
    \[ \eta = \sqrt{\mu/\varepsilon}, \quad E_{x0} = \eta J_{s0}/2 \]
Plane waves in free space (2)

• Plane waves solutions are similar to TL solutions!
  – Frequency domain expressions
    \[ \tilde{E}(z) = \hat{x} \left( \tilde{E}_x^+ e^{-jkz} + \tilde{E}_x^- e^{jkz} \right) \]
    \[ \tilde{H}(z) = \hat{y} \left( \frac{\tilde{E}_x^+}{\eta} e^{-jkz} - \frac{\tilde{E}_x^-}{\eta} e^{jkz} \right) \]
  – Time domain expressions
    \[ E(t, z) = \hat{x} \left( |\tilde{E}_x^+| \cos(\omega t - kz + \phi^+) + |\tilde{E}_x^-| \cos(\omega t + kz + \phi^-) \right) \]
    \[ H(t, z) = \hat{y} \left( \frac{|\tilde{E}_x^+|}{\eta} \cos(\omega t - kz + \phi^+) - \frac{|\tilde{E}_x^-|}{\eta} \cos(\omega t + kz + \phi^-) \right) \]
  – Parameters
    ▪ Phase velocity \( v_p = 1/\sqrt{\mu \varepsilon} = \omega/k \) \( (1/\sqrt{\mu_0 \varepsilon_0} = 2 \times 10^8) \)
    ▪ Wavelength \( \lambda = 2\pi/k = v_p / f \)
    ▪ Characteristic (intrinsic) impedance \( \eta = \sqrt{\mu / \varepsilon} \) \( (\eta_0 = \sqrt{\mu_0 / \varepsilon_0} = 120\pi \Omega) \)
Plane waves in free space (3)

- Equivalence between plane waves and TL waves
  - Replace $\tilde{V} \rightarrow \tilde{E}, I \rightarrow H, Z_0 \rightarrow \eta$
  - The resulting waves will have the same behavior

$$\tilde{I} = \frac{1}{Z_0} \tilde{V}$$

$$\tilde{H}_y = \frac{1}{\eta} \tilde{E}_x$$
Plane waves in free space (4)

• Relation between $\tilde{E}$ & $\tilde{H}$ and propagating direction

  – Let $\hat{k} = \hat{x}\sin{\theta}\cos{\phi} + \hat{y}\sin{\theta}\sin{\phi} + \hat{z}\cos{\theta}$ be the propagation direction of a plane wave with $\theta$ & $\phi$ being angles

  – General solutions

    $\tilde{E} = \tilde{E}_0 e^{-jk\hat{k}\cdot r}$, $\tilde{H} = \tilde{H}_0 e^{-jk\hat{k}\cdot r}$

  – Field-direction relations

    \[
    \tilde{H}_0 = \frac{1}{\eta} \hat{k} \times \tilde{E}_0 \\
    \tilde{E}_0 = -\eta \hat{k} \times \tilde{H}_0
    \]

    or

    \[
    \tilde{H} = \frac{1}{\eta} \hat{k} \times \tilde{E} \\
    \tilde{E} = -\eta \hat{k} \times \tilde{H}
    \]

  – Plane waves are TEM waves $\tilde{E} \perp \hat{k}$, $\tilde{H} \perp \hat{k}$, $\tilde{E} \perp \tilde{H}$

  – $\tilde{E}, \tilde{H}, \hat{k}$ form a right-handed triple

  – Plane waves satisfy Maxwell’s and Helmholtz equations
Plane waves in free space (5)

- **Examples**
  - \( \hat{k} = \hat{z}, \quad \tilde{E}_x = \tilde{E}_x^+ e^{-jkz} \Rightarrow \tilde{H}_y = \tilde{E}_x / \eta \)
  - \( \hat{k} = \hat{z}, \quad \tilde{E}_y = \tilde{E}_y^+ e^{-jkz} \Rightarrow \tilde{H}_x = -\tilde{E}_y / \eta \)
  - Any combination of the two above
  - Parallel polarization (TM field)
    \( \hat{k} = \hat{x} \sin \theta + \hat{z} \cos \theta, \quad \tilde{H} = \tilde{H}_y^+ e^{-jk(x \sin \theta + z \cos \theta)} \)
    \( \Rightarrow \tilde{E} = \eta \tilde{H}_y^+ (\hat{x} \cos \theta - \hat{z} \sin \theta) e^{-jk(x \sin \theta + z \cos \theta)} \)
  - Perpendicular polarization (TE field)
    \( \hat{k} = \hat{x} \sin \theta + \hat{z} \cos \theta, \quad \tilde{E} = \tilde{E}_y^+ e^{-jk(x \sin \theta + z \cos \theta)} \)
    \( \Rightarrow \tilde{H} = \frac{\tilde{E}_y^+}{\eta} (-\hat{x} \cos \theta + \hat{z} \sin \theta) e^{-jk(x \sin \theta + z \cos \theta)} \)
Plane waves in free space (6)

• Plane waves in lossy media
  
  - \( \varepsilon \rightarrow \varepsilon_c = \varepsilon - j \frac{\sigma}{\omega} = \varepsilon' + j\varepsilon'' \)
  
  - \( \Rightarrow \) wave equation \( \nabla^2 \tilde{E} - \gamma^2 \tilde{E} = 0 \)

  \[ \gamma = \alpha + j\beta \]

  1. Consider \( \mathbf{k} = \hat{z}, \tilde{E} = \hat{x}\tilde{E}_x \)

  \[ \Rightarrow |\tilde{E}_x(z)| = |E_{x0}e^{-\alpha z}e^{-j\beta z}| = |E_{x0}|e^{-\alpha z} \]

  - Skin depth \( \delta_s = \frac{1}{\alpha} \) (m)

  - Dielectrics vs. conductors

  - \( \varepsilon''/\varepsilon' \ll 1 \) – good dielectric (large \( \delta_s \))
  
  - \( \varepsilon''/\varepsilon' \gg 1 \) – good conductor (small \( \delta_s \))

Figure 7.13: Attenuation of the magnitude of \( \tilde{E}_x(z) \) with distance \( z \). The skin depth \( \delta_s \) is the value of \( z \) at which \( |\tilde{E}_x(z)|/|E_{x0}| = e^{-1} \), or \( z = \delta_s = 1/\alpha \).
Plane waves in free space (7)

- **Power flow (1)**
  - Poynting vector
    \[ S = \mathbf{E} \times \mathbf{H} \quad \text{(W/m}^2\text{)} \]
    - S is the power density (power per unit area)
    - It shows the strength AND direction of power flow
  - Power through a surface
    \[ P = \int_A S \cdot \hat{n} dA \quad \text{(W)} \]
  - Average power density
    \[ S_{av} = \frac{1}{2} \Re \left[ \mathbf{E} \times \mathbf{H}^* \right] \quad \text{(W/m}^2\text{)} \]
  - This is analogous to TLs!
    \[ P(z, t) = \nu(z, t) i(z, t) \]
    \[ P_{av}(z) = \frac{1}{2} \Re \left[ \mathbf{\nu}(z) \mathbf{I}^*(z) \right] \]
Plane waves in free space (8)

- **Power flow (2)**
  - Example: Plane waves in a lossless medium

\[
\vec{E}(z) = \hat{x} \vec{E}_x(z) + \hat{y} \vec{E}_y(z) \\
= (\hat{x} E_{x0} + \hat{y} E_{y0}) e^{-j k z},
\]

\[
\vec{H}(z) = \frac{1}{\eta} \hat{z} \times \vec{E} = \frac{1}{\eta} (-\hat{x} E_{y0} + \hat{y} E_{x0}) e^{-j k z}
\]

\[
\Rightarrow S_{av} = \frac{1}{2 \eta} (|E_{x0}|^2 + |E_{y0}|^2)
\]

\[
= \frac{\hat{z}}{2 \eta} \frac{(|\vec{E}|)^2}{(W/m^2)},
\]

\[
|\vec{E}| = (\vec{E} \cdot \vec{E}^*)^{1/2} = [|E_{x0}|^2 + |E_{y0}|^2]^{1/2}.
\]
Plane waves in free space (9)

- Why are plane waves (PWs) important?
  - PWs are simple solutions of Maxwell’s equations allowing learning many important wave properties
  - Fields radiated by antennas are local plane waves
  - Plane waves allow “canonical solution” of several important problems, e.g. reflection from an interface
  - Any field/current can be represented (expanded) as an integral/summation of a set of plane waves
  - Plane wave representations allow for solving many important problems
Plane waves at boundaries (1)

• Motivation
  – Problems involving boundaries are met in our every day life
  – Boundaries can lead to interference problems and we need to know their effects
  – Boundaries can be used to guide EM fields along them, e.g. fibers
  – There are many applications of wave phenomena occurring on boundaries
  – Problem of plane wave scattering from a planar boundary can be solved analytically
Plane waves at boundaries (2)

• Normal incidence (1)
  – We have already established a close similarity between plane wave and TL waves
  – This similarity can be extended to the problem of plane wave scattering from an interface
Plane waves at boundaries (3)

- Normal incidence (2)

**Incident Wave**

\[ \vec{E}^i(z) = \hat{x} E^i_0 e^{-jk_1 z}, \]

\[ \vec{H}^i(z) = \hat{z} \times \frac{\vec{E}^i(z)}{\eta_1} = \hat{y} \frac{E^i_0}{\eta_1} e^{-jk_1 z}. \]

**Reflected Wave**

\[ \vec{E}^r(z) = \hat{x} E^r_0 e^{jk_1 z}, \]

\[ \vec{H}^r(z) = (-\hat{z}) \times \frac{\vec{E}^r(z)}{\eta_1} = -\hat{y} \frac{E^r_0}{\eta_1} e^{jk_1 z}. \]

**Transmitted Wave**

\[ \vec{E}^t(z) = \hat{x} E^t_0 e^{-jk_2 z}, \]

\[ \vec{H}^t(z) = \hat{z} \times \frac{\vec{E}^t(z)}{\eta_2} = \hat{y} \frac{E^t_0}{\eta_2} e^{-jk_2 z}. \]

\[ k_1 = \omega \sqrt{\mu_1 \varepsilon_1} \quad k_2 = \omega \sqrt{\mu_2 \varepsilon_2} \]

\[ \eta_1 = \sqrt{\mu_1 / \varepsilon_1} \quad \eta_2 = \sqrt{\mu_2 / \varepsilon_2} \]
Plane waves at boundaries (4)

- Normal incidence (3)
  - From TL equivalence the reflection coefficient
    \[ \Gamma = \frac{E_0^r}{E_0^i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \]
  - From the boundary condition \( E_0^i + E_0^r = E_0^t \) and from the TL analogy, the transmission coefficient is given by
    \[ 1 + \Gamma = \tau \Rightarrow \tau = \frac{E_0^t}{E_0^i} = \frac{2\eta_2}{\eta_2 + \eta_1} \]
  - For non-magnetic media \( \mu_1 = \mu_2 = \mu_0 \)
    \[ \eta_1 = \frac{\eta_0}{\sqrt{\varepsilon_1}}, \eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_2}} \Rightarrow \Gamma = \frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}, \tau = \frac{2\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} \]
Plane waves at boundaries (5)

- Normal incidence (4)
  - All result for standing waves in TLs apply here as well!
  - Standing wave ratio
    \[ S = \frac{|\tilde{E}_1|_{\text{max}}}{|\tilde{E}_1|_{\text{min}}} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \]
  - For matched media \( \eta_1 = \eta_2 \Rightarrow \Gamma = 0, S = 1 \)
  - For a PEC wall \( \eta_2 = 0 \Rightarrow \Gamma = -1, S = \infty \)
  - Electric field maxima
    \[-z = l_{\text{max}} = \frac{\theta_t + 2n\pi}{2k_1} = \frac{\theta_t\lambda_1}{4\pi} + \frac{n\lambda_1}{2} \]
    \[ n = 1, 2, \ldots, \text{ if } \theta_t < 0, \]
    \[ n = 0, 1, 2, \ldots, \text{ if } \theta_t \geq 0, \]
  - Electric field minima
    \[ l_{\text{min}} = \begin{cases} 
    l_{\text{max}} + \lambda_1/4, & \text{if } l_{\text{max}} < \lambda_1/4, \\
    l_{\text{max}} - \lambda_1/4, & \text{if } l_{\text{max}} \geq \lambda_1/4.
    \end{cases} \]
Plane waves at boundaries (6)

- Normal incidence (5): TL Analogy

\[
\begin{align*}
\vec{E}_1(z) &= \hat{x}E_0^i(e^{-jk_1z} + \Gamma e^{jk_1z}) \quad (8.11a) \\
\vec{H}_1(z) &= \hat{y} \frac{E_0^i}{\eta_1}(e^{-jk_1z} - \Gamma e^{jk_1z}) \quad (8.12a) \\
\vec{E}_2(z) &= \hat{x}\tau E_0^i e^{-jk_2z} \quad (8.13a) \\
\vec{H}_2(z) &= \hat{y}\tau \frac{E_0^i}{\eta_2} e^{-jk_2z} \quad (8.14a) \\
\Gamma &= (\eta_2 - \eta_1)/(\eta_2 + \eta_1) \\
\tau &= 1 + \Gamma \\
k_1 &= \omega \sqrt{\mu_1\varepsilon_1}, \quad k_2 = \omega \sqrt{\mu_2\varepsilon_2} \\
\eta_1 &= \sqrt{\mu_1/\varepsilon_1}, \quad \eta_2 = \sqrt{\mu_2/\varepsilon_2}
\end{align*}
\]

\[
\begin{align*}
\vec{V}_1(z) &= V_0^+(e^{-j\beta_1z} + \Gamma e^{j\beta_1z}) \quad (8.11b) \\
\vec{I}_1(z) &= \frac{V_0^+}{Z_{01}}(e^{-j\beta_1z} - \Gamma e^{j\beta_1z}) \quad (8.12b) \\
\vec{V}_2(z) &= \tau V_0^+ e^{-j\beta_2z} \quad (8.13b) \\
\vec{I}_2(z) &= \tau \frac{V_0^+}{Z_{02}} e^{-j\beta_2z} \quad (8.14b) \\
\Gamma &= (Z_{02} - Z_{01})/(Z_{02} + Z_{01}) \\
\tau &= 1 + \Gamma \\
\beta_1 &= \omega \sqrt{\mu_1\varepsilon_1}, \quad \beta_2 = \omega \sqrt{\mu_2\varepsilon_2} \\
Z_{01} \text{ and } Z_{02} \text{ depend on transmission-line parameters}
\end{align*}
\]
Plane waves at boundaries (7)

- Normal incidence (6): Power flow
  - Net average power flow in the first medium
    \[ S_{av1}(z) = \frac{1}{2} \text{Re}[\vec{E}_1(z) \times \vec{H}^*_1(z)] = \frac{1}{2} \text{Re} \left[ \hat{x} E_0^i(e^{-jk_1z} + \Gamma e^{jk_1z}) \times \hat{y} \frac{E_0^{i*}}{\eta_1} (e^{jk_1z} - \Gamma^* e^{-jk_1z}) \right] \]
    \[ = \hat{z} \frac{|E_0^i|^2}{2\eta_1} (1 - |\Gamma|^2) \]
    \[ \Rightarrow S_{av1} = S_{av}^i + S_{av}^r; \quad S_{av}^i = \hat{z} \frac{|E_0^i|^2}{2\eta_1}; \quad S_{av}^r = -\hat{z}|\Gamma|^2 \frac{|E_0^i|^2}{2\eta_1} = -|\Gamma|^2 S_{av}^i \]
  - Net average power flow in the second medium
    \[ S_{av2}(z) = \frac{1}{2} \text{Re}[\vec{E}_2(z) \times \vec{H}^*_2(z)] = \frac{1}{2} \text{Re} \left[ \hat{x} \tau E_0^i e^{-jk_2z} \times \hat{y} \tau^* \frac{E_0^{i*}}{\eta_2} e^{jk_2z} \right] \]
    \[ = \hat{z}|\tau|^2 \frac{|E_0^i|^2}{2\eta_2} \]
  - Power relations
    \[ |S_{av1}| = |S_{av2}| \Rightarrow \boxed{\frac{\tau^2}{\eta_2} = \frac{1 - \Gamma^2}{\eta_1}} \]
Plane waves at boundaries (8)

- Oblique incidence (1)
  - What is different?
    - There are field components normal and tangential to the interface
    - Reflection and transmission depend on the state of polarization of the incident field
    - A general polarization is written as a sum of parallel (TM) and perpendicular (TE) polarizations
    - Reflecting properties for the parallel and perpendicular polarizations are studied separately
    - Angle of transmission is different from angle of incidence
Plane waves at boundaries (9)

• Oblique incidence (2): Snell’s law

  – Consider a plane wave at an incidence angle $\theta_i$
    \[
    \tilde{E}^i = \tilde{E}_0 e^{-jk_1(x\sin\theta_i + z\cos\theta_i)} = \tilde{E}_0 e^{-jk_1x\sin\theta_i} e^{-jk_1z\cos\theta_i}
    \]
  – The reflected and transmitted fields are
    \[
    \tilde{E}^r = \tilde{E}_0 e^{-jk_1x\sin\theta_r} e^{jk_1z\cos\theta_r} ; \quad \tilde{E}^t = \tilde{E}_0 e^{-jk_2x\sin\theta_t} e^{-jk_2z\cos\theta_t}
    \]
  – The boundary conditions at $z=0$
    \[
    \hat{z} \times (\tilde{E}^i + \tilde{E}^r) = \hat{z} \times \tilde{E}^t \Rightarrow \hat{z} \times \left( \tilde{E}_0 e^{-jk_1x\sin\theta_i} + \tilde{E}_0 e^{-jk_1x\sin\theta_r} \right) = \hat{z} \times \tilde{E}_0 e^{-jk_2x\sin\theta_t}
    \]
  – To satisfy the boundary conditions, the phase along the boundary has to be matched!
    \[
    e^{jk_1x\sin\theta_i} = e^{jk_1x\sin\theta_r} = e^{jk_2x\sin\theta_t}
    \]
Plane waves at boundaries (10)

- Oblique incidence (3): Snell’s law (cont’d)
  - From the phase matching condition
    \[ k_1 \sin \theta_i = k_1 \sin \theta_r = k_2 \sin \theta_t \]
  - Define indices of refraction \( n_1 = k_1 / k_0, n_2 = k_2 / k_0 \)
  - Snell’s law (general form)
    \[
    n_1 \sin \theta_i = n_1 \sin \theta_r = n_2 \sin \theta_t
    \]
  - Snell’s law of reflection
    \[ \theta_i = \theta_r \]
  - Snell’s law of refraction
    \[
    \frac{\sin \theta_t}{\sin \theta_i} = \frac{n_1}{n_2} = \frac{u_{p2}}{u_{p1}}
    \]
Plane waves at boundaries (11)

• Oblique incidence (4): Snell’s law (cont’d)
  – Transmission from and into a dense medium
    - Denser media have larger $n$
    - Transmission into a dense medium
      $n_2 > n_1 \Rightarrow \theta_t < \theta_i$
    - Incidence from a dense medium
      $n_2 < n_1 \Rightarrow \theta_t > \theta_i$
  – Total internal reflection (to be continued…)
    - $\theta_t = \frac{\pi}{2} \Rightarrow \sin \theta_c = \frac{n_2}{n_1} \sin \theta_t|_{\theta_t=\pi/2} = \frac{n_2}{n_1}$
    - $\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$ - critical angle
    - $\theta_i > \theta_c$ - total internal reflection (no field can propagate in the second medium; further discussion will follow shortly)
Plane waves at boundaries (12)

- Oblique incidence (5): Perpendicular polarization
  - Incident field
    \[
    \vec{E}_\perp^i = \hat{y}\vec{E}_{\perp 0}e^{-jk_1(x \sin \theta_i + z \cos \theta_i)}
    \]
    \[
    \vec{H}_\perp^i = (\hat{x} \cos \theta_i + \hat{z} \sin \theta_i)\frac{\vec{E}_{\perp 0}}{\eta_1}e^{-jk_1(x \sin \theta_i + z \cos \theta_i)}
    \]
  - Reflected field
    \[
    \vec{E}_\perp^r = \hat{y}\vec{E}_{\perp 0}e^{-jk_1(x \sin \theta_r + z \cos \theta_r)}
    \]
    \[
    \vec{H}_\perp^r = (\hat{x} \cos \theta_r + \hat{z} \sin \theta_r)\frac{\vec{E}_{\perp 0}}{\eta_1}e^{-jk_1(x \sin \theta_r - z \cos \theta_r)}
    \]
  - Transmitted field
    \[
    \vec{E}_\perp^t = \hat{y}\vec{E}_{\perp 0}e^{-jk_2(x \sin \theta_t + z \cos \theta_t)}
    \]
    \[
    \vec{H}_\perp^t = (-\hat{x} \cos \theta_t + \hat{z} \sin \theta_t)\frac{\vec{E}_{\perp 0}}{\eta_2}e^{-jk_2(x \sin \theta_t + z \cos \theta_t)}
    \]
Plane waves at boundaries (13)

- Oblique incidence (6): Perpendicular polarization
  - Boundary conditions
    \[
    \hat{z} \times (\vec{E}_\perp^i + \vec{E}_\perp^r) = \hat{z} \times \vec{E}_\perp^t
    \]
    \[
    \hat{z} \times (\vec{H}_\perp^i + \vec{H}_\perp^r) = \hat{z} \times \vec{H}_\perp^t
    \]
    \[
    \vec{E}_\perp y^i + \vec{E}_\perp y^r = \vec{E}_\perp y^t \Rightarrow \vec{E}_\perp^i 0 + \vec{E}_\perp^r 0 = \vec{E}_\perp^t 0
    \]
    \[
    \Rightarrow \vec{H}_\perp x^i + \vec{H}_\perp x^r = \vec{H}_\perp x^t \Rightarrow \frac{-\vec{E}_\perp^i 0 \cos \theta_i}{\eta_1} + \frac{\vec{E}_\perp^r 0 \cos \theta_r}{\eta_1} = \frac{-\vec{E}_\perp^t 0 \cos \theta_t}{\eta_2}
    \]
  - Solve and obtain the coefficients!
    \[
    \Gamma_\perp = \frac{E_{\perp 0}^r}{E_{\perp 0}^i} = \frac{\eta_2}{\cos \theta_i} - \frac{\eta_1}{\cos \theta_i}
    \]
    \[
    \tau_\perp = \frac{E_{\perp 0}^t}{E_{\perp 0}^i} = 1 + \Gamma_\perp = \frac{2}{\cos \theta_i} \frac{\eta_2}{\eta_1 + \eta_1} + \frac{\eta_1}{\cos \theta_i}
    \]
Plane waves at boundaries (14)

- Oblique incidence (7): Parallel polarization
  - Incident field
    \[
    \tilde{E}^i_\parallel = (\hat{x}\cos \theta_i - \hat{z}\sin \theta_i) \tilde{E}^i_\parallel 0 e^{-jk_1 (x \sin \theta_i + z \cos \theta_i)}
    \]
    \[
    \tilde{H}^i_\parallel = \hat{y} \frac{\tilde{E}^i_\parallel 0}{\eta_1} e^{-jk_1 (x \sin \theta_i + z \cos \theta_i)}
    \]
  - Reflected field
    \[
    \tilde{E}^r_\parallel = (\hat{x}\cos \theta_r + \hat{z}\sin \theta_r) \tilde{E}^r_\parallel 0 e^{-jk_1 (x \sin \theta_r - z \cos \theta_r)}
    \]
    \[
    \tilde{H}^r_\parallel = -\hat{y} \frac{\tilde{E}^r_\parallel 0}{\eta_1} e^{-jk_1 (x \sin \theta_r + z \cos \theta_r)}
    \]
  - Transmitted field
    \[
    \tilde{E}^t_\parallel = (\hat{x}\cos \theta_t - \hat{z}\sin \theta_t) \tilde{E}^t_\parallel 0 e^{-jk_2 (x \sin \theta_t + z \cos \theta_t)}
    \]
    \[
    \tilde{H}^t_\parallel = \hat{y} \frac{\tilde{E}^t_\parallel 0}{\eta_2} e^{-jk_2 (x \sin \theta_t + z \cos \theta_t)}
    \]

The sign is chosen to keep the same x component of the electric field!
Plane waves at boundaries (15)

- Oblique incidence (8): Parallel polarization
  - Boundary conditions
    \[ \mathbf{z} \times (\mathbf{E}^i + \mathbf{E}^r) = \mathbf{z} \times \mathbf{E}^t \]
    \[ \mathbf{z} \times (\mathbf{H}^i + \mathbf{H}^r) = \mathbf{z} \times \mathbf{H}^t \]
    \[ \mathbf{E}^i_x + \mathbf{E}^r_x = \mathbf{E}^t_x \Rightarrow \mathbf{E}^i_0 \cos \theta_i + \mathbf{E}^r_0 \cos \theta_r = \mathbf{E}^t_0 \cos \theta_r \]
    \[ \Rightarrow \mathbf{H}^i_y + \mathbf{H}^r_y = \mathbf{H}^t_y \Rightarrow \frac{\mathbf{E}^i_0}{\eta_1} - \frac{\mathbf{E}^r_0}{\eta_1} = \frac{\mathbf{E}^t_0}{\eta_2} \]
  - Solve and obtain the coefficients!

\[ \Gamma = \frac{E^r}{E^t} = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i} \]
\[ \tau = \frac{E^t}{E^i} = 1 + \Gamma = \frac{2 \eta_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i} \]
Plane waves at boundaries (16)

- Oblique incidence (9): TL equivalence
  - Observations
    - Boundary conditions are given in terms of tangential components
    - Tangential components depend on the incidence angle and polarization
  - Equivalence
    - Define the “characteristic” impedance via the tangential components
    
    \[
    \eta_{\perp,1,2} = \frac{\vec{E}_{\perp,\tan}^{i,t}}{\vec{H}_{\perp,\tan}^{i,t}} = \frac{\vec{H}_{\perp,x}^{i,t}}{\tilde{H}_{\perp,0}^{i,t} \cos \theta_{i,t}} = \eta_{1,2} \cos \theta_{i,t}
    \]
    
    \[
    \eta_{\parallel,1,2} = \frac{\vec{E}_{\parallel,x}^{i,t}}{\tilde{H}_{\parallel,y}^{i,t}} = \frac{\tilde{E}_{\parallel,0}^{i,t}}{\tilde{H}_{\parallel,0}^{i,t}} = \eta_{1,2} \cos \theta_{i,t}
    \]

    - Replace \( Z_{01} \& Z_{02} \) in a TL junction by \( \eta_{\perp,1} \& \eta_{\perp,2} \) for the perpendicular and \( \eta_{\parallel,1} \& \eta_{\parallel,2} \) for the parallel polarizations
    - Differences: \( e^{-jk_1x \sin \theta_i} \) phase propagation identical for ALL fields
Plane waves at boundaries (17)

- Oblique incidence (10): TL equivalence (cont’d)
  - Unified expressions
    \[ \Gamma_{\perp,||} = \frac{\eta_\perp,||2 - \eta_\perp,||1}{\eta_\perp,||2 + \eta_\perp,||1} \]
    \[ \tau_{\perp,||} = \frac{2\eta_\perp,||2}{\eta_\perp,||2 + \eta_\perp,||1} \]
  - Particular cases
    - Perpendicular polarization
      \[ \eta_{\perp,1,2} = \frac{\eta_{1,2}}{\cos \theta_{i,t}} \implies \Gamma_{\perp} = \frac{\eta_2 - \eta_1}{\eta_2 \cos \theta_{t} + \eta_1 \cos \theta_{i}} \]
      \[ \tau_{\perp} = \frac{2\eta_2}{\eta_2 \cos \theta_{t} + \eta_1 \cos \theta_{i}} \]
    - Parallel polarization
      \[ \eta_{||,1,2} = \eta_{1,2} \cos \theta_{i,t} \implies \Gamma_{||} = \frac{\eta_2 \cos \theta_{t} - \eta_1 \cos \theta_{i}}{\eta_2 \cos \theta_{t} + \eta_1 \cos \theta_{i}} \]
      \[ \tau_{||} = \frac{2\eta_2 \cos \theta_{t}}{\eta_2 \cos \theta_{t} + \eta_1 \cos \theta_{i}} \]
Plane waves at boundaries (18)

- **Oblique incidence (11): Properties**
  - Coefficients depend on the polarization (for a general polarization, the field is written as a sum of two polarizations with different properties)
  - The coefficient depends on the angle of incidence
  - The coefficients do not depend on the frequency for lossless case
  - Depending on the angle, the coefficients can be real or COMPLEX even in the lossless case!
Plane waves at boundaries (19)

- **Oblique incidence (12): Brewster angle**
  - Consider the case of matched impedance
    \[ \eta_2 \perp = \eta_1 \perp \Rightarrow \Gamma_\perp = 0 \]
    \[ \eta_2 \parallel = \eta_1 \parallel \Rightarrow \Gamma_\parallel = 0 \]
  - This is obtained under the Brewster angle
    - \[ \perp: \quad \frac{\eta_2}{\cos \theta_i} = \frac{\eta_1}{\cos \theta_i} \Rightarrow \sin \theta_B \perp = \sqrt{\frac{1 - \mu_1 \varepsilon_2 / (\mu_2 \varepsilon_1)}{1 - (\mu_2 / \mu_1)^2}} \]
      for non-magnetic materials \( \mu_1 = \mu_2 = \mu_0 \), \( \theta_B, \perp \) does not exist!
    - ||: \[ \eta_2 \cos \theta_i = \eta_1 \cos \theta_i \Rightarrow \sin \theta_B \parallel = \sqrt{\frac{1 - \varepsilon_1 \mu_2 / (\varepsilon_2 \mu_1)}{1 - (\varepsilon_1 / \varepsilon_2)^2}} \]
      for non-magnetic materials \( \theta_B \parallel = \tan^{-1} \sqrt{\varepsilon_2 / \varepsilon_1} \)
Plane waves at boundaries (20)

- **Oblique incidence (13): Total internal reflection**
  - Incidence from a denser medium $n_1 > n_2$ (e.g. from water to air)
  - Critical angle $\theta_i = \theta_c = \sin^{-1}(n_2/n_1) \Rightarrow \theta_t = \pi/2$
  - $\theta_i > \theta_c \Rightarrow \cos \theta_t = \pm j |\cos \theta_t|$
    $\Rightarrow \eta_\perp 2 & \eta_\perp 2$ are purely reactive (loads)!!!
  - $\Rightarrow \Gamma_\perp = e^{j\phi_\perp}, \Gamma_\parallel = e^{j\phi_\parallel}$ - phase shift at the interface
    $|\Gamma_\perp| = |\Gamma_\parallel| = 1$ - total internal reflection!!!
    $\phi_\perp = \phi_\perp(\theta_i), \ \phi_\parallel = \phi_\parallel(\theta_i)$ - angular dependence!!!
    $\phi_\perp \neq \phi_\parallel$ - dependence on polarization!!!
Plane waves at boundaries (21)

- Oblique incidence (14):
  - The phenomenon of total internal reflection is used to make fibers!

Figure 8-12: Waves can be guided along optical fibers as long as the reflection angles exceed the critical angle for total internal reflection.
Plane waves at boundaries (22)

- Oblique incidence (15): Summary

<table>
<thead>
<tr>
<th>Property</th>
<th>Normal Incidence $\theta_i = \theta_t = 0$</th>
<th>Perpendicular Polarization</th>
<th>Parallel Polarization</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reflection coefficient</td>
<td>$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$</td>
<td>$\Gamma_\perp = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_t}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t}$</td>
<td>$\Gamma_\parallel = \frac{\eta_2 \cos \theta_t - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i}$</td>
</tr>
<tr>
<td>Transmission coefficient</td>
<td>$\tau = \frac{2\eta_2}{\eta_2 + \eta_1}$</td>
<td>$\tau_\perp = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t}$</td>
<td>$\tau_\parallel = \frac{2\eta_2 \cos \theta_t}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i}$</td>
</tr>
<tr>
<td>Relation of $\Gamma$ to $\tau$</td>
<td>$\tau = 1 + \Gamma$</td>
<td>$\tau_\perp = 1 + \Gamma_\perp$</td>
<td>$\tau_\parallel = (1 + \Gamma_\parallel) \frac{\cos \theta_i}{\cos \theta_t}$</td>
</tr>
<tr>
<td>Reflectivity</td>
<td>$R =</td>
<td>\Gamma</td>
<td>^2$</td>
</tr>
<tr>
<td>Transmissivity</td>
<td>$T =</td>
<td>\tau</td>
<td>^2 \left(\frac{\eta_1}{\eta_2}\right)$</td>
</tr>
<tr>
<td>Relation of $R$ to $T$</td>
<td>$T = 1 - R$</td>
<td>$T_\perp = 1 - R_\perp$</td>
<td>$T_\parallel = 1 - R_\parallel$</td>
</tr>
</tbody>
</table>

Notes: (1) $\sin \theta_t = \sqrt{\mu_1 \varepsilon_1 / \mu_2 \varepsilon_2} \sin \theta_i$; (2) $\eta_1 = \sqrt{\mu_1 / \varepsilon_1}$; (3) $\eta_2 = \sqrt{\mu_2 / \varepsilon_2}$; (4) for nonmagnetic media, $\eta_2 / \eta_1 = n_1 / n_2$. 