Uniqueness Theorem

A given physical situation always leads to one and only one physical solution. However, when formulated in mathematical terms, if not properly done, the problem may lead to many acceptable solutions with prescribed boundary conditions, or it may permit no solutions at all with overprescribed boundary conditions. The uniqueness theorem states how a problem should be properly formulated mathematically so that there is one and only one solution. For electromagnetic problems, it states that when the sources and the tangential electric or magnetic fields are prescribed over the whole boundary surface of a given region, then the solution within this region is unique. The uniqueness theorem is thus a most powerful theorem that enables one to find the solution via any expedient means. It is the foundation for the equivalence principle, the Huygens’ principle, the image theorem, induction theorem, Babinet’s principle, and almost all frequently used methods in electromagnetism.

To prove the uniqueness theorem, we assume that there are two different solutions for a given set of sources. Let the two solutions be denoted by \( \mathbf{E}_1 \) and \( \mathbf{H}_1 \), and \( \mathbf{E}_2 \) and \( \mathbf{H}_2 \). Let the differences be \( \delta \mathbf{E} = \mathbf{E}_1 - \mathbf{E}_2 \)

\[
\delta \mathbf{H} = \mathbf{H}_1 - \mathbf{H}_2
\]

These field solutions satisfy Maxwell’s equations with the same sources, their differences satisfy the source-free Maxwell’s equations:

\[
\nabla \times \delta \mathbf{E} = i \omega \mu \delta \mathbf{H}
\]

\[
\nabla \times \delta \mathbf{H} = -i \omega \varepsilon \delta \mathbf{E}
\]

The proof of the theorem hinges on the assumption that the permittivity \( \varepsilon \) and the permeability \( \mu \) of the medium have a small imaginary part. Assume the medium is slightly lossy, namely \( \mu \) and \( \varepsilon \) have a small positive imaginary part,

\[
\varepsilon = \varepsilon_R + i \varepsilon_I
\]

\[
\mu = \mu_R + i \mu_I
\]

where \( \varepsilon_R, \mu_R \) and \( \mu_I \) are real. The proof also holds when the imaginary parts are both negative.

Dot-multiply (3a) with \( \delta \mathbf{H}^* \) and (3b) with \( \delta \mathbf{E} \). Subtracting, we obtain

\[
\nabla \cdot (\delta \mathbf{E} \times \delta \mathbf{H}^*) = i \omega \mu |\delta \mathbf{H}|^2 - i \omega \varepsilon |\delta \mathbf{E}|^2
\]

(4)

The complex conjugate of (4) gives

\[
\nabla \cdot (\delta \mathbf{E}^* \times \delta \mathbf{H}) = -i \omega \mu^* |\delta \mathbf{H}|^2 + i \omega \varepsilon |\delta \mathbf{E}|^2
\]

(5)

Adding (4) and (5) and integrating over the surface \( S \) enclosing the region \( V \), we find

\[
\iint_S dS \cdot (\delta \mathbf{E} \times \delta \mathbf{H}^* + \delta \mathbf{E}^* \times \delta \mathbf{H})
\]

\[
= -2\omega \iiint_V dV \left( \mu_I |\delta \mathbf{H}|^2 + \varepsilon_I |\delta \mathbf{E}|^2 \right)
\]

(6)

The right-hand side of (6) is a negative number. It will be zero if and only if \( \delta \mathbf{H} \) and \( \delta \mathbf{E} \) are identically zero in the region \( V \).

The solution will be unique and \( \mathbf{E}_1 = \mathbf{E}_2 \) and \( \mathbf{H}_1 = \mathbf{H}_2 \) when the left-hand side of (6) is zero, which gives

\[
\iint_S dS \cdot (\delta \mathbf{E} \times \delta \mathbf{H}^* + \delta \mathbf{E}^* \times \delta \mathbf{H}) = 0
\]

We conclude that the solution will be unique if either \( \delta \mathbf{E} \) or \( \delta \mathbf{H} \) is zero on the enclosed surface \( S \). Thus the boundary conditions can be specified in the following manner: (i) tangential electric field over the whole surface of \( S \), or (ii) tangential magnetic field over the whole surface of \( S \), or (iii) tangential electric field over part of the surface \( S \) and tangential magnetic field over the rest of \( S \). If both tangential electric and magnetic fields are specified over any part of \( S \), they must be compatible with each other.

5.2 Duality and Complementarity

Corresponding to the electric current \( \mathbf{J} \) in Ampere’s law, a magnetic current \( -\mathbf{M} \) can be added to Faraday’s law. Maxwell’s equations with both the electric and magnetic current terms read:

\[
\nabla \times \mathbf{H} = -i \omega \varepsilon \mathbf{E} + \mathbf{J}
\]

(1)

\[
\nabla \times \mathbf{E} = i \omega \mu \mathbf{H} - \mathbf{M}
\]

(2)
The justification of the magnetic current $M$ has been carried out with the use of the equivalence principle and is reiterated as follows. First, (1) and (2) govern macroscopic electromagnetic fields under time-harmonic excitations. From the macroscopic point of view, a small current loop acts like a magnetic dipole. As long as one is restricted from penetrating the loop, the fields outside the volume bounding the loop are exactly the dual of those due to a small electric dipole. The fields can be solved in exactly the same manner as in the electric case by neglecting $\mathbf{J}$ and retaining $\mathbf{M}$. Second, when (1) and (2) are applied to a limited region in space, the bounding surfaces of the regions can be viewed as supporting surface electric currents due to discontinuities in tangential magnetic field and surface magnetic currents due to discontinuities in tangential electric field. In fact, in the above discussion of dipole sources, we are limited to the space outside the volume occupied by the dipoles. If (1) and (2) are assumed to be valid in all space, then the existence of magnetic monopoles is implied.

Equations (1) and (2) are now duals of each other. If we make the following replacements:

\[
\begin{align*}
\mathbf{E} &\rightarrow \mathbf{H} \\
\mathbf{H} &\rightarrow -\mathbf{E} \\
\mu &\rightarrow \varepsilon \\
\varepsilon &\rightarrow \mu \\
\mathbf{J} &\rightarrow \mathbf{M} \\
\mathbf{M} &\rightarrow -\mathbf{J}
\end{align*}
\]

then (1) becomes (2) and (2) becomes (1). The symbolic replacements can further be quantified by equating numerically the left-hand side and the right-hand side of the arrows. When such equalities are established, however, the dual of free space will become a medium with permittivity $4\pi \times 10^{-7}$ f/m and with permeability $8.854 \times 10^{-12}$ h/m, an undesirable situation for the study of antenna problems.

The duality principle that is suitable for antenna and radiation problems in free space is established by the following equalities:

\[
\begin{align*}
\mathbf{E} &= \eta \mathbf{H} \\
\mathbf{H} &= -\mathbf{E}/\eta \\
\mathbf{J} &= \mathbf{M}/\eta \\
\mathbf{M} &= -\eta \mathbf{J}
\end{align*}
\]

where $\eta = (\mu/\varepsilon)^{1/2}$. Clearly under such substitutions, (1) becomes (2) and (2) becomes (1). There is no need to replace free space with a different medium. Note that this duality formalism is not applicable to complicated material such as anisotropic and bianisotropic media.

As an example of the application of the duality principle, we shall find a relationship between impedances of complementary metal and aperture antennas. We consider a plane conductor that is cut to make a metal antenna and an aperture antenna complementary to each other as shown in Figure 5.2.1. The input impedance of the metal antenna in Figure 5.2.1a is

\[
Z_m = -\frac{\int_{c}^{d} dl \cdot \mathbf{E}_m}{\int_{c}^{d} dl \cdot \mathbf{H}_m} = \frac{-\int_{c}^{d} dl \cdot \mathbf{E}_m}{2 \int_{c}^{d} dl \cdot \mathbf{H}_m}
\]

Figure 5.2.1 a. Metal antenna, and b. aperture antenna, which are complementary structures.

(7)

where we have assumed that $\mathbf{E}_m$ is pointing from $a$ to $b$ and $\mathbf{H}_m$ from $c$ to $d$. The second equality follows from the fact that tangential magnetic fields are equal and opposite on both sides of the path $c$ to $d$.

Similarly, the input impedance of the aperture antenna is

\[
Z_a = -\frac{\int_{c}^{d} dl \cdot \mathbf{E}_a}{\int_{c}^{d} dl \cdot \mathbf{H}_a} = \frac{\int_{c}^{d} dl \cdot \mathbf{E}_a}{2 \int_{c}^{d} dl \cdot \mathbf{H}_a}
\]

(8)
where we assume the dual situation with $\bar{H}_a$ pointing from $a$ to $b$ and $\bar{E}_a$ from $d$ to $c$.

The two impedances are related by duality properties of Maxwell's equations. The boundary-value problem for the metal antenna is to solve

$$(\nabla^2 + \omega^2 \mu \varepsilon) \bar{E}_m = 0$$

with boundary conditions

$$\hat{n} \times \bar{E}_m = 0 \quad \text{on} \quad S_a$$
$$\hat{n} \times \bar{H}_m = 0 \quad \text{on} \quad S_m$$

where

$$\bar{H}_m = \frac{1}{i \omega \mu} \nabla \times \bar{E}_m$$

and $\hat{n}$ is the normal to the plane conductor. Note that the second boundary condition in (10) can be understood by imagining that the surface currents on the metal are composed of elementary current sources with magnetic field having only components perpendicular to $S_m$.

For the aperture antenna problem, we have to solve

$$(\nabla^2 + \omega^2 \mu \varepsilon) \bar{H}_a = 0$$

with boundary conditions

$$\hat{n} \times \bar{H}_a = 0 \quad \text{on} \quad S_a$$
$$\hat{n} \times \bar{E}_a = 0 \quad \text{on} \quad S_m$$

where

$$\bar{E}_a = -\frac{1}{i \omega \varepsilon} \nabla \times H_a$$

The two problems are mathematically dual with the following replacements:

$$\bar{E}_m = \eta \bar{H}_a$$
$$\bar{H}_m = -\frac{1}{\eta} \bar{E}_a$$

We find from (7) and (8)

$$Z_m Z_a = \frac{\int_b^a \eta \bar{H}_a \cdot d\bar{l}}{-2 \int_c^d \frac{\bar{E}_a \cdot d\bar{l}}{\eta}} \left( \frac{\int_c^d \bar{E}_a \cdot d\bar{l}}{2 \int_b^a \bar{H}_a \cdot d\bar{l}} \right) = \frac{\eta^2}{4}$$

Thus, the product of the input impedances of two planar complementary antennas is one-quarter of the square of the characteristic impedance of the free space.

As an example, consider the structures shown in Figures 5.2.2, where the structure and its complement are identical. It follows that such antennas have input impedance $\eta/2 = 188.5$ ohms. Because the input impedance is independent of frequency, such antennas are ideal broadband antennas. Another self-complementary antenna is shown in Figure 5.2.3. The four edges of the metal are described by the equations $\rho_1 = \rho_0 e^{i \phi}$, $\rho_2 = \rho_0 e^{i (\phi - \pi/2)}$, $\rho_3 = \rho_0 e^{i (\phi - \pi)}$ and $\rho_4 = \rho_0 e^{i (\phi - 3\pi/2)}$, where $1/a$ is the rate of expansion of the spiral. The structure is known as a planar equiangular spiral antenna.

Babinet's principle is another example of duality and complementarity that relates the problem of diffraction by planar apertures to the problem of scattering by its complementary structure. Consider an infinite plane conductor with an aperture as shown in Figure 5.2.5a and the complementary structure consisting of the removed plane metal in the formation of the aperture as shown in Figure 5.2.5b. Let there be dual sources on the left-hand sides of Figures 5.2.5a and 5.2.5b. In the absence of the screens, they produce incident fields such that

$$\bar{E}_2 = \eta \bar{H}_1$$
Figure 5.2.3 Planar equiangular spiral antenna.

\[ \overline{H}_2^i = -\overline{E}_1^i/\eta \]  
(18b)

Note that in the presence of the screens the sources are located on the left-hand side.

Now we formulate the problems for the fields on the right-hand sides of the screens. For the problem in Figure 5.2.4a, the fields satisfy Maxwell's equations

\[ \nabla \times \overline{E}_1 = i\omega \mu \overline{H}_1 \]  
(19a)
\[ \nabla \times \overline{H}_1 = -i\omega \varepsilon \overline{E}_1 \]  
(19b)

subject to the boundary conditions

\[ \hat{n} \times \overline{E}_1 = 0 \quad \text{on} \quad S_m \]  
(20a)
\[ \hat{n} \times \overline{H}_1 = \hat{n} \times \overline{H}_1^i \quad \text{on} \quad S_a \]  
(20b)

where \( \hat{n} \) is the normal to the plane surface. The first boundary condition warrants that tangential electric field vanishes on the conducting surface. The second condition is due to the fact that induced surface currents on the metal surface produce no tangential magnetic field component at the aperture space. The field to the right-hand side of

Figure 5.2.4 Complementarity and duality for Babinet's principle.
the screen is produced by the equivalent current sheet source \( \mathcal{J}_1 \) on \( S_a \).

For the problem in Figure 5.2.4b, the fields on the right-hand side satisfy Maxwell's equations

\[
\nabla \times \mathcal{E}_2 = i \omega \mu \mathcal{H}_2
\]
(21a)
\[
\nabla \times \mathcal{H}_2 = -i \omega \epsilon \mathcal{E}_2
\]
(21b)

subject to the boundary conditions

\[
\hat{n} \times \mathcal{H}_2 = \hat{n} \times \mathcal{H}_2^i \quad \text{on} \quad S_m
\]
(22a)
\[
\hat{n} \times \mathcal{E}_2 = 0 \quad \text{on} \quad S_a
\]
(22b)

The total fields \( \mathcal{E}_2 \) and \( \mathcal{H}_2 \) are superpositions of two field solutions:

1. The incident fields \( \mathcal{E}_2^i \) and \( \mathcal{H}_2^i \) in the absence of the screens. They satisfy source-free Maxwell's equations to the right half-space.
2. The scattered fields produced by induced currents on the screens. We denote them by \( \mathcal{E}_2^s \) and \( \mathcal{H}_2^s \).

In terms of the scattered field \( \mathcal{E}_2^s = \mathcal{E}_2 - \mathcal{E}_2^i \) and \( \mathcal{H}_2^s = \mathcal{H}_2 - \mathcal{H}_2^i \) they satisfy

\[
\nabla \times \mathcal{E}_2^s = i \omega \mu \mathcal{H}_2^s
\]
(23a)
\[
\nabla \times \mathcal{H}_2^s = -i \omega \epsilon \mathcal{E}_2^s
\]
(23b)

subject to the boundary conditions

\[
\hat{n} \times \mathcal{H}_2^s = \hat{n} \times \mathcal{H}_2^i \quad \text{on} \quad S_m
\]
(24a)
\[
\hat{n} \times \mathcal{E}_2^s = -\hat{n} \times \mathcal{E}_2^i = -\hat{n} \times \eta \mathcal{H}_1^i \quad \text{on} \quad S_a
\]
(24b)

Comparing (23) with (19) and (24) with (20), we see that the two problems are mathematically dual with the following substitutions

\[
\mathcal{H}_2^s = \mathcal{E}_1^i/\eta
\]
(25a)
\[
\mathcal{E}_2^s = -\eta \mathcal{H}_1^i
\]
(25b)

In terms of total fields, we find

\[
\mathcal{E}_2 = \mathcal{E}_2^s + \mathcal{E}_2^i = \eta(-\mathcal{H}_1 + \mathcal{H}_1^i)
\]
(26a)
\[
\mathcal{H}_2 = \mathcal{H}_2^s + \mathcal{H}_2^i = \frac{1}{\eta}(\mathcal{E}_1 - \mathcal{E}_1^i)
\]
(26b)

This is referred to as Babinet's principle. Note that in the special case of no metallic screens for Figure 5.2.4a, \( \mathcal{E}_1 = \mathcal{E}_1^i \) and \( \mathcal{H}_1 = \mathcal{H}_1^i \). The complementary case is a complete metallic screen and the result is \( \mathcal{E}_2 = \mathcal{H}_2 = 0 \). Consider the other extreme when Figure 5.2.4a is completely metallic with no apertures, \( \mathcal{E}_1 = \mathcal{H}_1 = 0 \). Then the result for Figure 5.2.4b becomes \( \mathcal{E}_2 = \eta \mathcal{H}_1^i \) and \( \mathcal{H}_2 = -\mathcal{E}_1^i/\eta \), which are just the fields generated by the dual sources.

To examine further the implications of Babinet's principle, consider the dual problem of Figure 5.2.4b as illustrated in Figure 5.2.4c. The metallic aperture \( S_a \) is now replaced by a magnetic conductor and the sources are the dual of Figure 5.2.4c and therefore identical to those of Figure 5.2.4b. the boundary-value problem for the right-hand side of the magnetic conductor becomes

\[
\nabla \times \mathcal{E}_d = i \omega \mu \mathcal{H}_d
\]
(27a)
\[
\nabla \times \mathcal{H}_d = -i \omega \epsilon \mathcal{E}_d
\]
(27b)

subject to the boundary conditions

\[
\hat{n} \times \mathcal{E}_d = \hat{n} \times \mathcal{E}_1^i \quad \text{on} \quad S_m
\]
(28a)
\[
\hat{n} \times \mathcal{H}_d = 0 \quad \text{on} \quad S_a
\]
(28b)

where \( \mathcal{E}_d \) and \( \mathcal{H}_d \) denote fields of Figure 5.2.8. From (19)–(20) and (27)–(28) we see that the sums of the fields \( \mathcal{E} = \mathcal{E}_1 + \mathcal{E}_d \) and \( \mathcal{H} = \mathcal{H}_1 + \mathcal{H}_d \) satisfy the following boundary-value problem

\[
\nabla \times (\mathcal{E}_1 + \mathcal{E}_d) = i \omega \mu (\mathcal{H}_1 + \mathcal{H}_d)
\]
(29a)
\[
\nabla \times (\mathcal{H}_1 + \mathcal{H}_d) = -i \omega \epsilon (\mathcal{E}_1 + \mathcal{E}_d)
\]
(29b)

with the boundary conditions

\[
\hat{n} \times (\mathcal{E}_1 + \mathcal{E}_d) = \hat{n} \times \mathcal{E}_1^i \quad \text{on} \quad S_m
\]
(30a)
\[
\hat{n} \times (\mathcal{H}_1 + \mathcal{H}_d) = \hat{n} \times \mathcal{H}_1^i \quad \text{on} \quad S_a
\]
(30b)

Thus the tangential fields on \( S_m \) and \( S_a \) are identical to those of \( \mathcal{E}_1^i \) and \( \mathcal{H}_1^i \) and by the uniqueness theorem, we conclude that

\[
\mathcal{E}_1 + \mathcal{E}_d = \mathcal{E}_1^i
\]
(31a)
\[
\mathcal{H}_1 + \mathcal{H}_d = \mathcal{H}_1^i
\]
(31b)
also follows from Babinet's principle as expressed in (26) with substitution of \( \overline{E}_2 = \eta \overline{H}_d \) and \( \overline{H}_2 = -\overline{E}_d/\eta \) as required by the lity principle.

**Mathematical Formulations of Huygens' Principle**

Huygens' principle states that the field solution in a region \( V' \) is completely determined by the tangential fields specified over the sur-
face \( S' \) enclosing \( V' \) [Fig. 5.3.1]. Formulated in mathematical terms, Huygens' principle expresses fields at an observation point in terms of fields at the boundary surface. Consider a surface \( S' \) enclosing a radiating source. The electric and magnetic fields outside the surface \( S' \) be shown to be of the following forms:

\[
\overline{E}(\overline{r}) = \iint_{S'} dS' \left\{ i\omega \mu \overline{G}(\overline{r}, \overline{r}') \cdot [\hat{n} \times \overline{H}(\overline{r}')] \\
+ \nabla \times \overline{G}(\overline{r}, \overline{r}') \cdot [\hat{n} \times \overline{E}(\overline{r}')] \right\}
\]

\[
\overline{H}(\overline{r}) = \iiint_{S'} dS' \left\{ -i\omega \epsilon \overline{G}(\overline{r}, \overline{r}') \cdot [\hat{n} \times \overline{E}(\overline{r}')] \\
+ \nabla \times \overline{G}(\overline{r}, \overline{r}') \cdot [\hat{n} \times \overline{H}(\overline{r}')] \right\}
\]

where \( \hat{n} \) is the outward normal to the surface \( S' \). The dyadic Green's function \( \overline{G}(\overline{r}, \overline{r}') \) is given by

\[
\overline{G}(\overline{r}, \overline{r}') = \left[ \overline{T} + \frac{1}{k^2} \nabla \nabla \right] g(\overline{r}, \overline{r}')
\]

The scalar Green's function \( g(\overline{r}, \overline{r}') \) satisfies the Helmholtz equation

\[
(\nabla^2 + k^2)g(\overline{r}, \overline{r}') = -\delta(\overline{r} - \overline{r}')
\]

For three-dimensional problems, the scalar Green's function \( g(\overline{r}, \overline{r}') \) in isotropic media written in spherical coordinates is of the form

\[
g(\overline{r}, \overline{r}') = \frac{e^{ik|\overline{r} - \overline{r}'|}}{4\pi |\overline{r} - \overline{r}'|}
\]

For two-dimensional problems, the scalar Green's function for isotropic media written in cylindrical coordinates is of the form

\[
g(\rho, \rho') = \frac{i}{4} H_0^{(1)}(k |\rho - \rho'|)
\]

From Maxwell's equations, the governing equation for the electric field \( \overline{E}(\overline{r}) \) with a given current source \( \overline{J}(\overline{r}) \) is given by

\[
\nabla \times \nabla \times \overline{E}(\overline{r}) - k^2 \overline{E}(\overline{r}) = i\omega \mu \overline{J}(\overline{r})
\]

The solution of \( \overline{E}(\overline{r}) \) in terms of \( \overline{J}(\overline{r}) \) can be conveniently expressed in terms of the dyadic Green's function \( \overline{G}(\overline{r}, \overline{r}') \),

\[
\overline{E}(\overline{r}) = i\omega \mu \iiint d^3\overline{r}' \overline{G}(\overline{r}, \overline{r}') \cdot \overline{J}(\overline{r}')
\]

Notice that by means of the three-dimensional delta function \( \delta(\overline{r} - \overline{r}') \), we can write

\[
\overline{J}(\overline{r}) = \iiint d^3\overline{r}' \delta(\overline{r} - \overline{r}') \overline{T} \cdot \overline{J}(\overline{r}')
\]

where \( \overline{T} \) is the unit dyad. Substitution of (8) and (9) in (7) gives the following equation governing the dyadic Green's function \( \overline{G}(\overline{r}, \overline{r}') \),

\[
\nabla \times \nabla \times \overline{G}(\overline{r}, \overline{r}') - k^2 \overline{G}(\overline{r}, \overline{r}') = \overline{T} \delta(\overline{r} - \overline{r}')
\]